② Cancellation of gauge anomalies in the SM.

Let's verify that the anomaly coefficient associated to any combination of SM currents is zero.

First of all, note that $\bar{\Psi}\gamma^\mu\gamma_5\Psi = \bar{\Psi}\gamma^\mu(P_R - P_L)\Psi$ thus

$$J_5^\mu = Q_R\,\bar{\Psi}\gamma^\mu\gamma_5 P_R\Psi - Q_L\,\bar{\Psi}\gamma^\mu\gamma_5 P_L\Psi$$

therefore, we need to compute the trace over R.H. fields, <u>minus</u> the trace over L.H. fields.

- $[U(1)]^3$ $\quad\left[Y_L = -\frac{1}{2},\ Y_e = -1,\ Y_\nu = 0,\ Y_Q = \frac{1}{6},\ Y_u = \frac{2}{3},\ Y_d = -1/3\right]$

$$\partial_\mu J^\mu_Y = -\left(\sum_{RH} Y^3_{\psi_L} - \sum_{LH} Y^3_{\psi_R}\right)\cdot\frac{g'^2}{32\pi^2}\,\underbrace{B\cdot\tilde{B}}_{=(\epsilon\cdot B)^{\mu\nu}}$$

$$\propto n_{\text{families}}\times\Big[2\underbrace{Y_L^3}_{-1/2} - \underbrace{Y_{e_R}^3}_{-1} - \underbrace{Y_{\nu_R}^3}_{0} + 3\Big(2\underbrace{Y_Q^3}_{1/6} - \underbrace{Y_{u_R}^3}_{2/3} - \underbrace{Y_{d_R}^3}_{-1/3}\Big)\Big] = 0$$

- $SU(2)_L U(1)_Y^2$ & $SU(3)_c U(1)_Y^2$

$$\propto \mathrm{Tr}\left[T^a\cdot\{\mathbb{1},\mathbb{1}\}\right] = 2\,\mathrm{Tr}[T^a] = 0$$

• $\left[SU(2)_L\right]^3$

$$\propto \mathrm{Tr}\left[\tau^a \cdot \{\tau^b, \tau^c\}\right] = \mathrm{Tr}\left[\tau^a \cdot \frac{1}{2}\delta^{bc} \cdot \mathbb{1}\right] = 0$$

• $\left[SU(3)_c\right]^3$

$SU(3)_c$ is not chiral ($\psi_L \sim \psi_R$). thus

$$\sum_{\text{LEFT}} (\quad) - \sum_{\text{RIGHT}} (\quad) = 0$$

• $\left[SU(3)_c\right]^2 \times U(1)_Y$

$$\propto \mathrm{Tr}\left[Y_L \cdot \{T^a, T^b\}\right] - \mathrm{Tr}\left[Y_R \{\overbrace{T^a, T^b}^{\frac{1}{2}\delta^{ab}}\}\right]$$

$$\frac{1}{2}\delta^{ab}\,\mathrm{Tr}\left[Y_L - Y_R\right]_{\text{QUARKS}} =$$

$$= \frac{1}{2}\delta^{ab} \cdot 3_{\text{COLORS}} \cdot \left(2 \cdot \frac{1}{6} - \frac{2}{3} + \frac{1}{3}\right) \cdot n_{\text{families}} = 0$$

$\cdot \left[SU(2)_L\right]^2 \times U(1)_Y$

$$\propto \mathrm{tr}\left[Y_{LEFT}\{\tau^a,\tau^b\}\right] = \frac{1}{2}\delta^{ab}\,\mathrm{Tr}\left[Y_{LEFT}\right]_{Q,L}$$

$$= \frac{1}{2}\delta^{ab}\left(3\cdot 2\cdot \frac{1}{6} - 2\cdot\frac{1}{2}\right)\times n_{families} = 0$$

3) Global anomalies.

a) $\boxed{B \,\&\, L}$

B and L are anomalous under

$$SU(2)_L^2 \times U(1)_B \quad \text{and} \quad SU(2)_L^2 \times U(1)_L$$

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

$$1/2\,\delta^{ab}\cdot\sum_{LEFT} B = \frac{1}{2}\delta^{ab}\cdot 3\cdot 2\cdot 1/3 \times n_f = \delta^{ab} n_f \qquad 1/2\,\delta^{ab}\cdot\sum_{LEFT} L = \frac{1}{2}\delta^{ab}\cdot 2n_f = \delta^{ab} n_f$$

B and L have the same anomaly-coefficient, thus the combination B-L is <u>not</u> anomalous. ✓

b) $\pi^0 \to \gamma\gamma$

Considering 2-flavors QCD, in the limit of massless u, d quarks, the global symmetry is

$$SU(2)_L \times U(1)_L \times SU(2)_R \times U(1)_R$$

the Axial component $SU(2)_A$ is spontaneously broken, and pions are the 3 Goldstone bosons arising from the SSB.

From Goldstone theorem, since Q, defined as:

$$Q = \int d^3x\, J_0(x) = \int d^3x\, \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} \frac{\delta \phi}{\delta \alpha}$$

is a conserved charge, thus $[H, Q] = 0$. This implies

$$HQ|\Omega\rangle = [H,Q]|\Omega\rangle + QH|\Omega\rangle = E_0 \cdot Q|\Omega\rangle$$

therefore Q generates states degenerate with the ground state.

CONVENIENT NORMALIZATION

$$|\pi^a(\vec{p})\rangle = -\frac{2i}{f_\pi} \int d^3x\, e^{i\vec{p}\vec{x}} J_{0A}^a(\vec{x})|\Omega\rangle$$

AXIAL CURRENT

Multiplying by $\langle \pi^b(\vec{q})|$ and using the normalization
$\langle \pi^a(\vec{q})|\pi^b(\vec{p})\rangle = 2\delta^{ab} E_p (2\pi)^3 \delta^3(p-q)$, we finally get:
(integrating over $\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{y}}$)

$$\langle \pi^a(\vec{q})|\, J^b_{A0}(y)\, |\Omega\rangle = i\, E_q\, f_\pi\, e^{i\vec{q}\cdot\vec{y}}\, \delta^{ab}$$

generalizing

$$\langle \pi^a(\vec{q})|\, J^b_{A\mu}(y)\, |\Omega\rangle = i\, q_\mu\, f_\pi\, e^{-iq_\mu y^\mu}\, \delta^{ab}$$

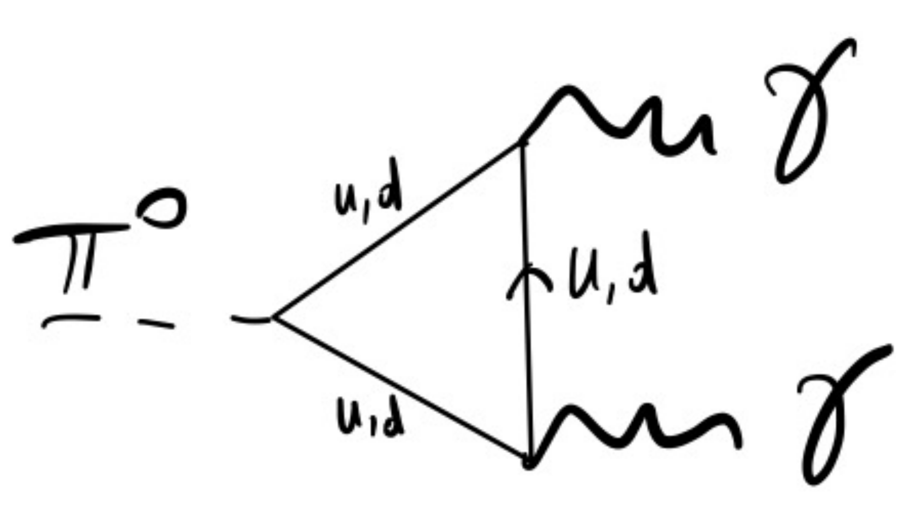
therefore
$$|\pi^0(q)\rangle \approx \frac{\partial_\mu J_A^{\mu 3}(x)}{q^2 f_\pi}\, e^{-iq\cdot x} \cdot |\Omega\rangle \qquad (*)$$

The $SU(2)_A$ symmetry is anomalous under $[U(1)_{EM}]^2$, since

$$3_{\text{colors}} \times \operatorname{tr}\Big[\underbrace{T^a}_{SU(2)\text{ gen.}} \cdot \underbrace{\{Q,Q\}}_{\text{quarks EM charges}}\Big] \overset{(a=3)}{=} \operatorname{tr}\left[\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot 2\begin{pmatrix}4/9 & 0\\ 0 & 1/9\end{pmatrix}\right]\times 3$$

$$= 3\left(4/9 - 1/9\right) = 1 \neq 0$$

thus



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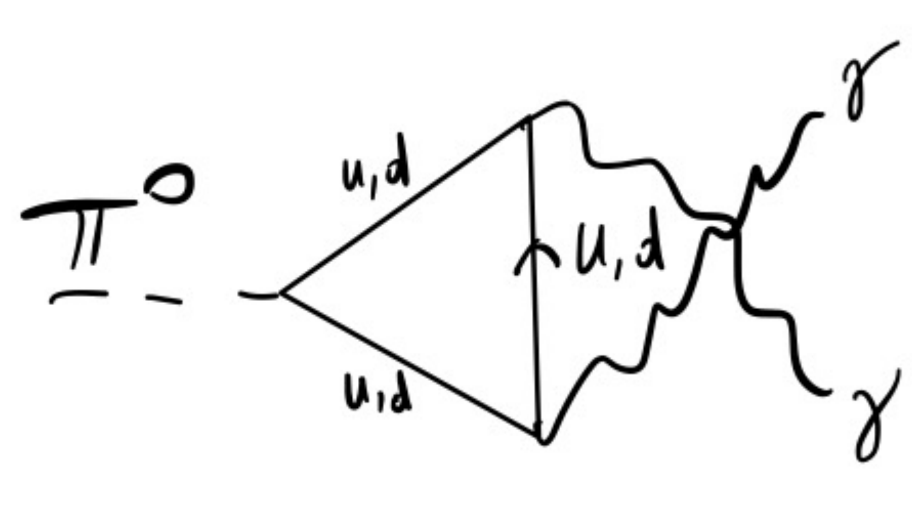

$\neq 0$,

and
$$\partial_\mu J_A^{\mu 3} = -\frac{e^2}{16\pi^2}\, \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

Let's compute the S matrix element $S_{\pi^0\to\gamma\gamma}$

$$S_{\pi^0\to\gamma\gamma} = (2\pi)^4 \delta^4(p-k_1-k_2) \cdot M(\pi^0\to\gamma\gamma) =$$

$$= \int d^4x\, d^4y\, d^4z\; e^{i(px-k_1y-k_2z)} \left(\Box_x + m_\pi^2\right)\left(\Box_y\right)\left(\Box_z\right) \times$$

$$\times \langle 0| T\{\pi^0(x)\, A^\mu(y)\, A^\nu(z)\}|0\rangle \cdot \epsilon_\mu^*(k_1)\, \epsilon_\nu^*(k_2) =$$

$$= \int d^4x\, d^4y\, d^4z\; e^{i(px-k_1y-k_2z)} \left(\Box_x + m_\pi^2\right) \times$$

$$\times \langle 0| T\{\pi^0(x)\, J^\mu(y)\, J^\nu(z)\}|0\rangle\, \epsilon_\mu^*(k_1)\, \epsilon_\nu^*(k_2) =$$

where we used the E.O.M. $\Box_x A^\mu(x) = J^\mu(x) = \bar\Psi(x)\gamma^\mu\Psi(x)$

Using Eq. $(*)$,

$$= \int d^4x\, d^4y\, d^4z\; e^{i(px-k_1y-k_2z)} \left(-\cancel{p^2} + \overset{\text{chiral limit}}{\cancel{m_\pi^2}}\right) \times$$

$$\times \langle 0| T\left\{ \frac{\partial_\alpha J_A^{3\alpha}(x)}{\cancel{p^2}\, f_\pi}\, e^{-ipx}\, J^\mu(y)\, J^\nu(z)\right\}|0\rangle\, \epsilon_\mu^*(k_1)\, \epsilon_\nu^*(k_2) =$$

$$= \frac{-\epsilon_\mu^{*1}\epsilon_\nu^{*2}}{f_\pi} \int d^4x\, d^4y\, d^4z\; \underbrace{e^{-ix(p-k_1-k_2)}}_{\to\,\delta(p-k_1-k_2)} e^{-ik_1(y-x)}\, e^{-ik_2(z-x)} \times$$

$$\times\; \partial_\alpha \langle J_A^{3\alpha}(0)\, J^\mu(y-x)\, J^\nu(z-x)\rangle$$

$$= \frac{-\epsilon_\mu^*(k_1)\,\epsilon_\nu^*(k_2)}{f_\pi} (2\pi)^4 \delta^4(p-k_1-k_2)\; \partial_\alpha \langle J_A^{3\alpha}(p)\, J^\mu(k_1)\, J^\nu(k_2)\rangle$$

$$= \underbrace{\boxed{+\frac{\epsilon_\mu^*(k_1)\,\epsilon_\nu^*(k_2)}{f_\pi}\cdot\frac{e^2}{4\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}\, k_1^\alpha\, k_2^\beta}}_{M_{\pi^0\to\gamma\gamma}} (2\pi)^4\, \delta^4(p-k_1-k_2)$$

Take the squared modulus:

$$|M|^2 = \frac{e^4}{(4\pi^2)^2 f_\pi^2} \varepsilon_{\mu\nu\alpha\beta}\, \varepsilon_{\gamma\delta\rho\sigma}\, \epsilon_1^{*\mu} \epsilon_2^{*\nu} \epsilon_1^{\gamma} \epsilon_2^{\delta} k_1^{\alpha} k_2^{\beta} k_1^{\rho} k_2^{\sigma}$$

using $\sum_{\lambda=1,2} \epsilon_\lambda^{\alpha}(k)\, \epsilon_\lambda^{*\beta}(k) = -g^{\alpha\beta} + \cancel{k^{\alpha,\beta}\text{ terms}}$

then

$$\sum_{\lambda_{1,2}} |M|^2 = \frac{e^4}{(4\pi^2)^2 f_\pi^2} \left(g^{\mu\gamma} \cdot g^{\nu\delta} \right) \cdot \varepsilon_{\mu\nu\alpha\beta}\, \varepsilon_{\gamma\delta\rho\sigma}\, k_1^{\alpha} k_2^{\beta} k_1^{\rho} k_2^{\sigma}$$

$$= \frac{\alpha^2}{f_\pi^2}\, 2\left(-g^{\alpha\rho} g^{\beta\sigma} + g^{\alpha\sigma} g^{\beta\rho}\right) k_1^{\alpha} k_2^{\beta} k_1^{\rho} k_2^{\sigma}$$

$$= \frac{\alpha^2}{f_\pi^2}\, 2\, (k_1 \cdot k_2)^2$$

$\hookrightarrow k_1 \cdot k_2 = \frac{1}{2}(k_1 + k_2)^2 = \frac{m_\pi^2}{2}$

$$= \frac{\alpha^2}{2\pi^2 f_\pi^2}\, m_\pi^4$$

therefore, for a 2-body decay:

$$\Gamma = \frac{1}{32\pi^2} |M|^2 \frac{m_\pi}{2} \frac{1}{m_\pi^2} \cdot 4\pi \cdot \frac{1}{2}\Big|_{\text{SYMM. FACT.}}$$

$$= \frac{1}{16\pi} \frac{\alpha^2}{2\pi^2 f_\pi^2} m_\pi^4 \cdot \frac{1}{2 m_\pi} = \boxed{\frac{\alpha^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2}}$$

✓