Vertiefungen zu Physik II - Woche 7

Mittwoch, 5. April 2023 06:52

Fourier-Reihe

$$f(x+L) = f(x) \qquad \text{Periode } L$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n x) + b_n \sin(k_n x) \qquad k_n = n\frac{2\pi}{L}$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos(k_n x)\,dx$$

$$= \frac{2}{L}\int_{-L/2}^{0} f(x)\cos(k_n x)\,dx + \frac{2}{L}\int_0^{L/2} f(x)\cos(k_n x)\,dx$$

$$\left(b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin(k_n x)\,dx \right)$$

$$= \frac{2}{L}\int_0^{L/2} f(-x)\cos(-k_n x)\,dx + \frac{2}{L}\int_0^{L/2} f(x)\cos(k_n x)\,dx$$

$$= \frac{2}{L}\int_0^{L/2} \big(f(x)+f(-x)\big)\cos(k_n x)\,dx$$

$$b_n = \text{analog} = \frac{2}{L}\int_0^{L/2} \big(f(x)-f(-x)\big)\sin(k_n x)\,dx$$

$\Rightarrow$ gerade Fn. $f(x) = f(-x) \Rightarrow$ nur cos-Terme
ungerade Fn $f(x) = -f(-x) \Rightarrow$ nur sin-Terme

$\Rightarrow$ komplexe Fourier-Reihe

$$\cos(k_n x) = \frac{1}{2}\left(e^{ik_n x} + e^{-ik_n x}\right) =: \frac{1}{2}\left(e^+ + e^-\right)$$

$$\sin(k_n x) = \frac{1}{2i}\left(e^{ik_n x} - e^{-ik_n x}\right) =: \frac{1}{2i}\left(e^+ - e^-\right)$$

$$f(x) = \frac{a_0}{2} + \sum \frac{a_n}{2}\left(e^+ + e^-\right) + \frac{b_n}{2i}\left(e^+ - e^-\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{+} + e^{-}\right) + \frac{b_n}{2i}\left(e^{+} - e^{-}\right)$$

$$= \frac{a_0}{2} e^{0} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - i\frac{b_n}{2}\right) e^{+} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + i\frac{b_n}{2}\right) e^{-}$$

$$= \underbrace{\frac{a_0}{2}}_{=:C_0} e^{0} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2} - i\frac{b_n}{2}\right)}_{\substack{=:C_n \\ n>0}} e^{ik_n x} + \sum_{n=-\infty}^{-1} \underbrace{\left(\frac{a_{-n}}{2} + i\frac{b_{-n}}{2}\right)}_{\substack{=:C_n \\ n<0}} e^{+ik_n x}$$

$$= \sum_{n=-\infty}^{\infty} C_n \, e^{ik_n x}$$

$$C_n \underset{n<0}{\overset{n\geq 0}{=}} \frac{a_n}{2} \mp i\frac{b_n}{2} = \frac{1}{2}\,\frac{2}{L}\int_{-L/2}^{L/2} f(x)\,\underline{\cos(k_n x)}\,dx \underline{\mp} i\,\frac{1}{2}\,\frac{2}{L}\int_{-L/2}^{L/2} f(x)\,\underset{-\sin(k_n x)}{\underline{\sin(k_n x)}}\,dx$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\left(\cos(k_n x) - i\sin(k_n x)\right) dx$$

$$\boxed{C_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-ik_n x}\, dx}$$

komplexe Fourier-Reihe

$C_n \in \mathbb{C}$

$$\Rightarrow \boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{ik_n x}}$$

Übergang $L \to \infty$ (nicht-periodische Fn.)

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{ik_n x}$$

$k_n = n \cdot \frac{2\pi}{L}$

$\underline{\Delta k} = \frac{2\pi}{L} = k_{n+1} - k_n$

$$= \frac{L}{2\pi} \sum C_n \, e^{ik_n x} \, \Delta k$$

$L \to \infty$

$$= \frac{L}{2\pi} \int \tilde{C}(k)\, e^{ikx}\, dk \qquad c(k) := L \cdot \tilde{c}(k)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k)\, e^{ikx}\, dk$$

$$c(k) = L \cdot \tilde{c}(k) = \cancel{L} \cdot \frac{1}{\cancel{L}} \underbrace{\int f(x)\, e^{-ikx}\, dx}_{\mathcal{F}[f] =: F(k)}$$

Fourier-Transformation einer beliebigen, stetigen Fn. $f(x)$

$$\boxed{\mathcal{F}[f] = F(k) := \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx}$$

inverse FT

$$\boxed{\mathcal{F}^{-1}[F(k)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{+ikx}\, dk}$$

(inverse) FT entspricht Basiswechsel zwischen

Realraum — Frequenzraum

Reziproker Raum

Ort $\vec{r}$ $\Longleftrightarrow$ $\vec{k}$ Wellenvektoren

$\vec{p} = \hbar \vec{k} = \frac{h}{2\pi} \cdot \vec{k}$ Impuls

reduziertes Planck'sches Wirkungsquantum

Zeit t $\Longleftrightarrow$ $\omega = 2\pi\nu$ Frequenzen

Vorfaktor $\frac{1}{2\pi}$

Hin- und Rücktransformation müssen $f(x)$ reproduzieren

1. $F(k) = \int f(x)\, e^{-ikx}\, dx$

$f(x) = \frac{1}{2\pi} \int F(k)\, e^{ikx}\, dk$

Problem: "Leistung" $\int \| f(x) \|^2 dx \neq \int \| F(k) \|^2 dk$

$\| f(x) \|^2 = \int f^*(x) f(x)\, dx$ $\mathcal{L}^2$- Norm

nicht erhalten!

2. $F(k) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{-ikx}\, dx$

$f(x) = \frac{1}{\sqrt{2\pi}} \int F(k)\, e^{ikx}\, dx$

Norm erhalten! (Physik)

3. $k \rightarrow y = \frac{1}{L} = \frac{k}{2\pi} \Rightarrow dy = \frac{dk}{2\pi}$

$\omega \rightarrow \nu = \frac{1}{T} = \frac{\omega}{2\pi} \Rightarrow d\nu = \frac{d\omega}{2\pi}$

$$\boxed{\begin{aligned} F(y) &= \int f(x)\, e^{-2\pi i y x}\, dx \\ f(x) &= \int F(y)\, e^{2\pi i x y}\, dy \end{aligned}}$$

Norm erhalten! (Technisch wichtig)

Unitäre Transformation (Satz von Plancherel)