Sheet 6, Ex. 3

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a.1) Under CP, $|K^0\rangle$ and $|\bar{K}^0\rangle$ transform as

$$CP|K^0\rangle = e^{i\xi}|\bar{K}^0\rangle$$

$$CP|\bar{K}^0\rangle = e^{-i\xi}|K^0\rangle$$

[CP is unitary and $(CP)^2 = \mathbb{1}$]

the phase ξ is unphysical, and we can fix $\boxed{e^{i\xi} = 1}$

the CP-eigenstates are

$$CP = + \qquad |K_1\rangle = \frac{1}{\sqrt{2}}\left(|K_0\rangle + |\bar{K}^0\rangle\right)$$

$$CP = - \qquad |K_2\rangle = \frac{1}{\sqrt{2}}\left(|K_0\rangle - |\bar{K}^0\rangle\right)$$

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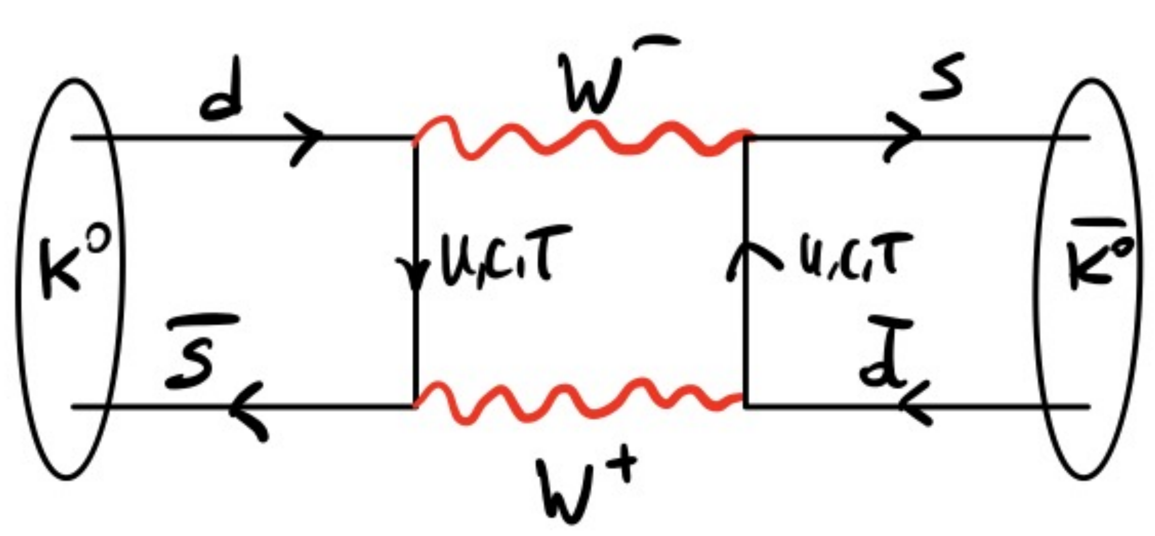
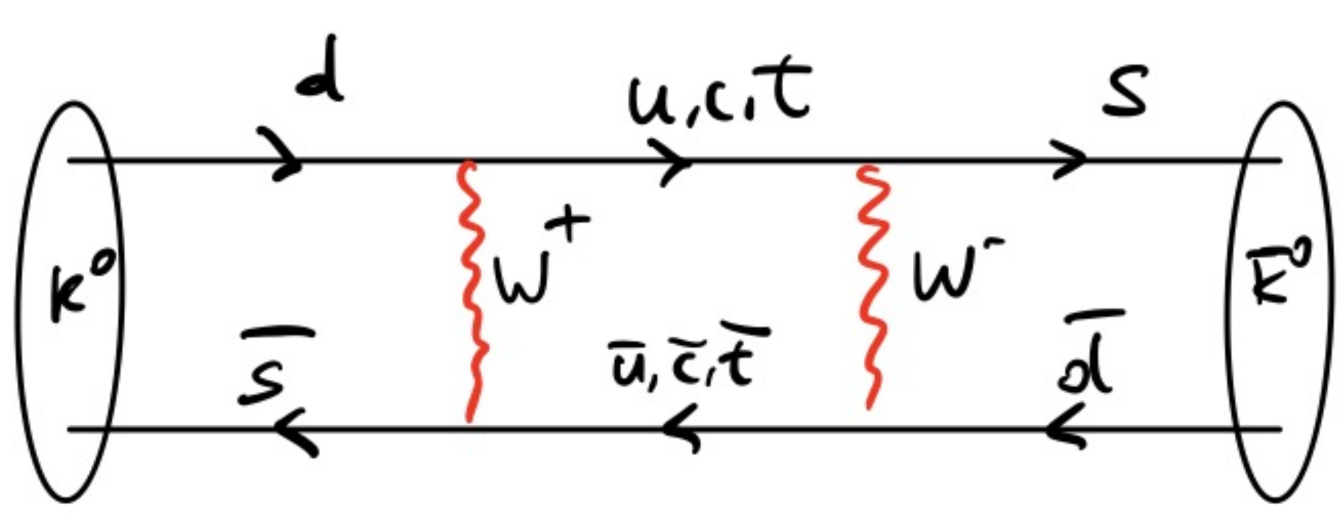
a.2) Weak interactions are responsible for the $K^0 - \bar{K}^0$ mixing, via the diagrams





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a.3) A beam of oscillating neutral mesons can be described by the 2-component wave-function

$$|\psi(t)\rangle = \psi_1(t)\,|K^0\rangle + \psi_2(t)\,|\bar{K}^0\rangle$$

the wave function evolves according to

$$i\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = H^{eff}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Since the kaons can decay, H is not Hermitian, and can be written as

$$H^{eff} = M - \frac{i}{2}\Gamma$$

where M and Γ are 2x2 hermitian matrices.

$$\text{Hermitian} \rightarrow \begin{cases} M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22}\end{pmatrix} \\ \Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{22}\end{pmatrix}\end{cases}, \quad M_{ii} \text{ and } \Gamma_{ii} \text{ REAL}$$

We have seen that under CP : $K^0 \leftrightarrow \bar{K}^0$, so

$U_{CP} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, while under T : $M \to M^*$, $\Gamma \to \Gamma^*$

Let's apply CPT on the matrix elements of M and Γ. ($M^{CPT} = M$, $\Gamma^{CPT} = \Gamma$)

- $M \to M' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} M_{22} & M_{12} \\ M_{12}^* & M_{11} \end{bmatrix} \overset{!}{=} M$

- $\Gamma \to \Gamma' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Gamma^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \Gamma_{22} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{11} \end{bmatrix} \overset{!}{=} \Gamma$

thus, CPT invariance on the matrix elements implies

$$M_{22} \overset{!}{=} M_{11} \equiv m_K , \qquad \Gamma_{22} \overset{!}{=} \Gamma_{11} \equiv \gamma$$

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b) the eigenvalues of $H^{eff} = M - \frac{i}{2}\Gamma$ are

$$M_{S,L} - \frac{i}{2}\Gamma_{S,L} \equiv \begin{cases} m_K - \frac{i}{2}\gamma + \overbrace{\sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)}}^{R} \\ m_K - \frac{i}{2}\gamma - \sqrt{\quad \text{''} \quad} \end{cases}$$

The mass and decay rates differences are:

$$\Delta m = M_S - M_L = 2\,\mathrm{Re}\,(R)$$

$$\Delta \Gamma = \Gamma_S - \Gamma_L = 4\,\mathrm{Im}\,(R)$$

The labels "S" and "L" denote the lifetimes of the eigenstates, since K_{SHORT} decays much faster than K_{LONG}.

The eigenstates are

$$|K_S\rangle = p|K^0\rangle + q\,|\bar{K}^0\rangle$$

$$|K_L\rangle = p|K^0\rangle - q\,|\bar{K}^0\rangle$$

Notice that p and q are defined up to a common phase, and

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}} \quad ; \quad |p^2| + |q^2| = 1$$

In the literature, the notation $q/p = \frac{1-\epsilon}{1+\epsilon}$ is often adopted, with $|\epsilon^{exp}| \sim 10^{-3}$.

Moreover, notice that

If: $\quad Im\left(M_{12}^{*} \Gamma_{12}\right) = 0$

then

$$\left.\begin{array}{l} |K_S\rangle = |K_1\rangle \\ |K_L\rangle = |K_2\rangle \end{array}\right\} \text{CP - eigenstates.}$$

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a.4) The time evolution of H^{eff} eigenstates is

$$|K_S(t)\rangle = e^{-i\left(M_S - \frac{i}{2}\Gamma_S\right)t}\,|K_S(0)\rangle$$

$$|K_L(t)\rangle = e^{-i\left(M_L - \frac{i}{2}\Gamma_L\right)t}\,|K_L(0)\rangle$$

Starting from a pure K^0 beam:

$$t=0: \quad |\psi(0)\rangle = |K^0\rangle = \frac{1}{2p}\left(|K_S(0)\rangle + |K_L(0)\rangle\right)$$

the beam evolves as:

$$|\psi(t)\rangle = \frac{1}{2p}\Big[e^{-i\left(M_S - \frac{i}{2}\Gamma_S\right)t}\cdot\left(p|K^0\rangle + q|\bar{K}^0\rangle\right) +$$

$$+ e^{-i\left(M_L - \frac{i}{2}\Gamma_L\right)t}\cdot\left(p|K^0\rangle - q|\bar{K}^0\rangle\right)\Big]$$

$$= f_+(t)\,|K^0\rangle + \frac{q}{p} f_-(t)\,|\bar{K}^0\rangle$$

where

$$f_\pm(t) = \frac{1}{2}\left[e^{-i(M_S - i/2\,\Gamma_S)t} \pm e^{-i(M_L - i/2\,\Gamma_L)t}\right]$$

the probability of finding a K^0 or $\bar{K}^0$ at time t are

$$N(K^0) = |\langle K^0|\psi(t)\rangle|^2 = |f_+(t)|^2$$

$$= \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2\,\mathrm{Re}\left(e^{-i\left(M_S - M_L - i/2(\Gamma_S+\Gamma_L)\right)t}\right)\right]$$

$$= \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2\, e^{-\frac{\Gamma_S+\Gamma_L}{2}t} \cdot \cos(\Delta M \cdot t)\right]$$

and

$$N(\bar{K}^0) = |\langle \bar{K}^0|\psi(t)\rangle|^2 = \left|\frac{q}{p}\right|^2 |f_-(t)|^2 =$$

$$= \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\cos(\Delta M t)\right] \cdot \left|\frac{q}{p}\right|^2$$

therefore the asymmetry $A(t) = \dfrac{N(K^0) - N(\bar{K}^0)}{N(K^0) + N(\bar{K}^0)}$ reads

$$A(t) = \frac{\left[\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right)\left(1 - \left|\frac{q}{p}\right|^2\right) + 2\, e^{-\frac{\Gamma_S+\Gamma_L}{2}t} \cdot \cos(\Delta M \cdot t)\left(1 + \left|\frac{q}{p}\right|^2\right)\right]}{\left[\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right)\left(1 + \left|\frac{q}{p}\right|^2\right) + 2\, e^{-\frac{\Gamma_S+\Gamma_L}{2}t} \cdot \cos(\Delta M \cdot t)\left(1 - \left|\frac{q}{p}\right|^2\right)\right]}$$

the beam oscillates with frequency $\Delta M / 2\pi$.
the oscillations are dumped by the K_S decays

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b) the K_S are the first to decay, because $\Gamma_S \gg \Gamma_L$, $c/\Gamma_S \sim 3\,cm$, $c/\Gamma_L \sim 15\,m$. thus, at $t \sim \frac{1}{\Gamma_L} \gg \frac{1}{\Gamma_S}$:

$$A\left(t \sim \frac{1}{\Gamma_L}\right) =$$

$$\frac{\left[\left(\cancel{e^{-\frac{\Gamma_S}{\Gamma_L}}}^{\,0} + e^{-1}\right)\left(1 - \left|\frac{q}{p}\right|^2\right) \cancel{+ 2\, e^{-\frac{\Gamma_S + \Gamma_L}{2}\frac{1}{\Gamma_L}} \cdot \cos\left(\Delta M / \Gamma_L\right)\left(1 + \left|\frac{q}{p}\right|^2\right)}\right]}{\left[\left(\cancel{e^{-\frac{\Gamma_S}{\Gamma_L}}} + e^{-1}\right)\left(1 + \left|\frac{q}{p}\right|^2\right) \cancel{+ 2\, e^{-\frac{\Gamma_S + \Gamma_L}{2}\frac{1}{\Gamma_L}} \cdot \cos\left(\Delta M / \Gamma_L\right)\left(1 - \left|\frac{q}{p}\right|^2\right)}\right]}$$

$$\Longrightarrow \quad A\left(t \sim \frac{1}{\Gamma_L}\right) \simeq \frac{1 - |q/p|^2}{1 + |q/p|^2}$$

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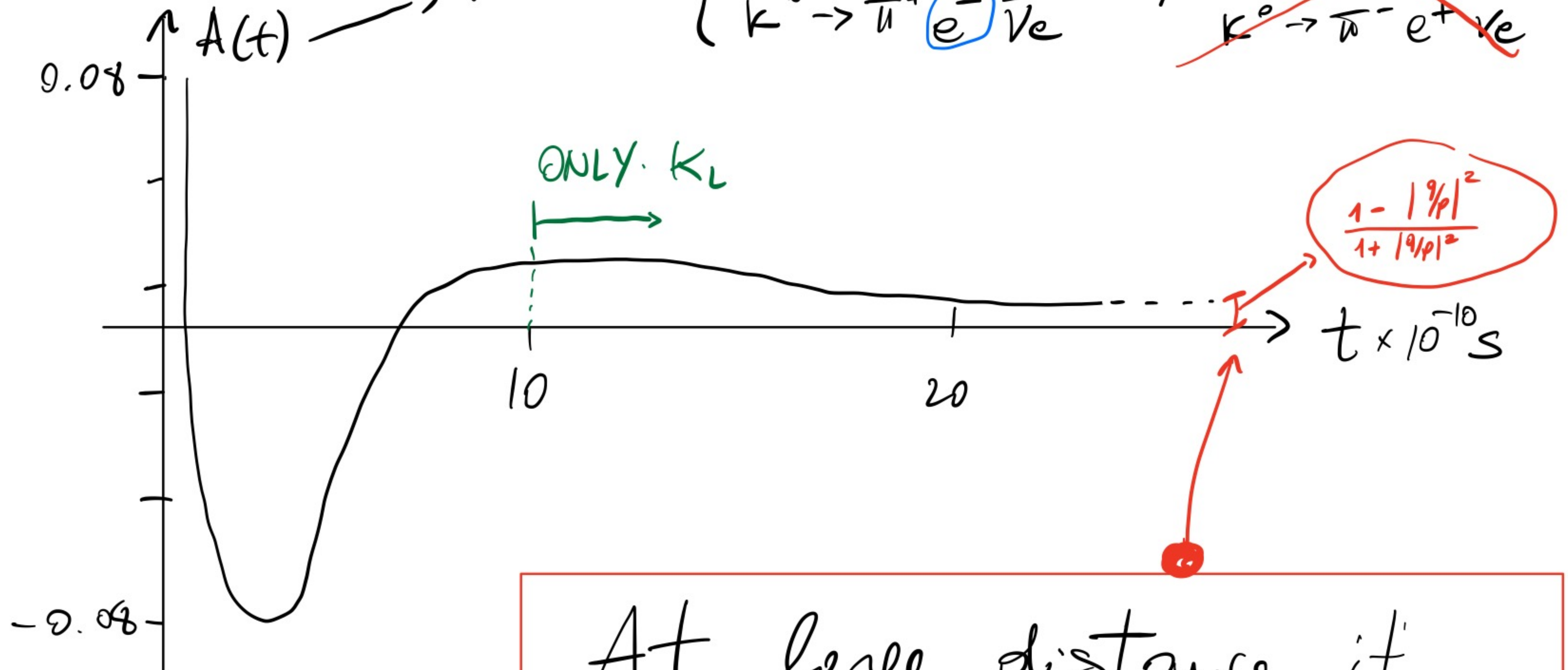
c). the Asymmetry $A(t)$ has this behavior:

$A(t) \rightarrow$ measure $\begin{cases} K^0 \to \pi^- e^+ \nu_e \\ \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e \end{cases}$, NOT ALLOWED $\cancel{\begin{matrix} \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e \\ K^0 \to \pi^- e^+ \nu_e \end{matrix}}$



At large distance it has been observed that $\pi^- e^+ \nu_e$ is 0.7% more likely than the decay $\pi^+ e^- \bar{\nu}_e$.

The fact that $|q/p|^2 \neq 1$ implies that K_S and K_L are different w.r.t. $K_{1,2}$, thus they are not CP eigenstates, and CP is violated.