Standard Model and Beyond
Sheet 6

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Exercise 1: Neutrino oscillations

In the Standard Model neutrinos are strictly massless, however the observation of neutrino oscillations clearly indicate that they are massive and that there is lepton flavour mixing. Hence neutrino the flavour eigenstates (ν_α with $\alpha = e, \mu, \tau$) are mixture of the massive neutrinos (ν_k with $k = 1, 2, 3$) such that

$$|\nu_\alpha\rangle = U^*_{\alpha k}\,|\nu_k\rangle\,, \tag{1}$$

where U is the unitary Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix, that can be parametrised as

$$U \;=\; \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\,e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\,e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\,e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\,e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\,e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \mathrm{diag}(1,\, e^{i\varphi_2},\, e^{i\varphi_3})\,, \tag{2}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, with θ_{ij} the neutrino mixing angles, δ denotes the Dirac CP violating phase and $\varphi_{2,3}$ are the CP violating Majorana phases (should neutrinos be Majorana fermions).

a) The massive neutrino states are eigenstates of the Hamiltonian, implying that

$$|\nu_k(t)\rangle = e^{-iE_k t}|\nu_k\rangle\,. \tag{3}$$

Show that the oscillation probability in vacuum for a neutrino of flavour α produced at $t = 0$ (i.e. $|\nu_\alpha(t=0)\rangle = |\nu_\alpha\rangle$) to be detected at a distance L with the flavour β can be expressed as

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta; L) = \delta_{\alpha\beta} \quad &- \quad 4\sum_{k>j} \Re[U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j}]\sin^2\left(\frac{\Delta m^2_{kj}L}{4E}\right) \\ &+ \quad 2\sum_{k>j} \Im[U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j}]\sin\quad\frac{\Delta m^2_{kj}L}{2E}\Bigg)\,, \end{aligned} \tag{4}$$

where $\Delta m^2_{kj} = m^2_k - m^2_j$.
Hint: we consider ultrarelativistic neutrinos, approximate that the propagation time t is equal to the distance travelled L and assume that all the massive neutrinos have the same momentum $|\vec{p_k}| = |\vec{p_j}| = E$.

b) From the appearance probability in Eq. 4 deduce the probability $P(\bar\nu_\alpha \to \bar\nu_\beta)$. What can you conclude ?

c) From the formulae derive above and the parametrisation of the PMNS matrix in Eq. 2, can we distinguish between Majorana and Dirac neutrino via the oscillation phenomenon?

Solution:

a) Considering the flavour state α created at $t = 0$, we have from Eqs. **??** and **??**

$$|\nu_\alpha(t)\rangle = \sum_k U^*_{\alpha k} e^{-iE_k t}|\nu_k\rangle \,, \tag{5}$$

then expressing the mass eigenstates in terms of the flavour one (inverting Eq. **??**), we have

$$|\nu_\alpha(t)\rangle = \sum_{\lambda=e,\mu,\tau} \left(\sum_k U^*_{\alpha k}\, e^{-iE_k t}\, U_{\lambda k} \right) |\nu_\lambda\rangle \,. \tag{6}$$

The amplitude of the transition $\nu_\alpha \to \nu_\beta$ is given by

$$\begin{aligned} A_{\nu_\alpha \to \nu_\beta}(t) \equiv \langle \nu_\beta | \nu_\alpha(t)\rangle &= \langle \nu_\beta | \sum_{\lambda,k} U^*_{\alpha k}\, e^{-iE_k t}\, U_{\lambda k} |\nu_\lambda\rangle \\ &= \sum_{\lambda,k} \left[U^*_{\alpha k}\, e^{-iE_k t}\, U_{\lambda k} \langle \nu_\beta | \nu_\lambda \rangle \right] \,, \end{aligned} \tag{7}$$

the mass and consequently also the flavour states being orthonormal ($\langle \nu_\beta | \nu_\lambda\rangle = \delta_{\beta\lambda}$), we have

$$A_{\nu_\alpha \to \nu_\beta}(t) = \sum_k U^*_{\alpha k}\, e^{-iE_k t}\, U_{\beta k} \,. \tag{8}$$

The transition probability is thus given by

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta; t) = |A_{\nu_\alpha \to \nu_\beta}(t)|^2 &= \left[\sum_k U^*_{\alpha k}\, e^{-iE_k t}\, U_{\beta k} \right] \left[\sum_j U_{\alpha j}\, e^{iE_j t}\, U^*_{\beta j} \right] \\ &= \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i(E_k - E_j)t} \,. \end{aligned} \tag{9}$$

Considering ultrarelativistic neutrinos lead to

$$E_k = \sqrt{\vec{p_k}^{\,2} + m_k^2} \approx |\vec{p_k}| \left(1 + \frac{m_k^2}{2|\vec{p_k}|^2} \right) \,, \tag{10}$$

using now the "equal momentum" assumption ($|\vec{p_k}| = |\vec{p_j}| = E$) the energy reads

$$E_k = E + \frac{m_k^2}{2E} \,, \tag{11}$$

giving the energy difference

$$E_k - E_j = \frac{m_k^2 - m_j^2}{2E} = \frac{\Delta m^2_{kj}}{2E} \,. \tag{12}$$

The probability is thus

$$P(\nu_\alpha \to \nu_\beta; t) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i \frac{\Delta m^2_{kj}}{2E} t \right) \,. \tag{13}$$

Finally in the light approximation (i.e. $L = t$ as neutrinos propagate almost at the speed of light and the time is not measured in experiment but the distance L is), the vacuum oscillation formula is given by

$$P(\nu_\alpha \to \nu_\beta; L, E) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i \frac{\Delta m^2_{kj}}{2E} L \right) \,. \tag{14}$$

This expression can be recast as

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta; L, E) &= \sum_{k=j} |U_{\alpha k}|^2 |U_{\beta k}|^2 + \sum_{k\neq j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i\frac{\Delta m^2_{kj}}{2E} L\right) \\
&= \sum_{k} |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\sum_{k>j} \Re\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i\frac{\Delta m^2_{kj}}{2E} L\right)\right] \\
&= \sum_{k} |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\sum_{k>j} \Re[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \cos\left(\frac{\Delta m^2_{kj}}{2E} L\right) \\
&\quad + 2\sum_{k>j} \Im[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \sin\left(\frac{\Delta m^2_{kj}}{2E} L\right) .
\end{aligned}
\tag{15}
$$

Using the unitary condition of the PMNS matrix

$$
\left(\sum_k U^*_{\alpha k} U_{\beta k}\right)\left(\sum_j U_{\alpha j} U^*_{\beta j}\right) = \sum_{k=j} |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\sum_{k>j} \Re[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] = \delta_{\alpha\beta} \, , \tag{16}
$$

one has

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta; L, E) &= \delta_{\alpha\beta} - 2\sum_{k>j} \Re[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \left[1 - \cos\left(\frac{\Delta m^2_{kj}}{2E} L\right)\right] \\
&\quad + 2\sum_{k>j} \Im[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \sin\left(\frac{\Delta m^2_{kj}}{2E} L\right) \\
&= \delta_{\alpha\beta} - 4\sum_{k>j} \Re[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \sin^2\left(\frac{\Delta m^2_{kj} L}{4E}\right) \\
&\quad + 2\sum_{k>j} \Im[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \sin\left(\frac{\Delta m^2_{kj} L}{2E}\right) ,
\end{aligned}
\tag{17}
$$

b) Starting from Eq. 4, one has to make the substitution $U \to U^*$ in order to obtain the antineutrino oscillation probability, leading to

$$
\begin{aligned}
P(\bar{\nu}_\alpha \to \bar{\nu}_\beta; L, E) &= \delta_{\alpha\beta} - 4\sum_{k>j} \Re[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}] \sin^2\left(\frac{\Delta m^2_{kj} L}{4E}\right) \\
&\quad + 2\sum_{k>j} \Im[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}] \sin\left(\frac{\Delta m^2_{kj} L}{2E}\right) .
\end{aligned}
\tag{18}
$$

Due to the Dirac CP violating phase δ present in the PMNS matrix

$$
\Im[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}] = -\Im[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \, , \tag{19}
$$

hence

$$
P(\nu_\alpha \to \nu_\beta; L, E) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta; L, E) = 4\sum_{k>j} \Im[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}] \sin\left(\frac{\Delta m^2_{kj} L}{2E}\right) \neq 0 \tag{20}
$$

implying that there is CP violation in the lepton sector (notice however that CPT is conserved $P(\nu_\alpha \to \nu_\beta; L, E) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha; L, E)$).

c) If one considers Majorana neutrinos, the mixing matrix is given by Eq.2, which is simply $U_{\alpha k} = U^{\text{Dirac}} e^{i\varphi_k}$. The quartic product $[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}]$ is invariant under rephasing, hence the Majorana phases (should neutrino be Majorana fermions) cannot be measured in oscillation experiments.

Implications of neutrino oscillations:
- massive and non-degenerated neutrinos,
- can probe Δm but not the absolute mass,
- the sign of Δm is not given by vacuum oscillations so we don't know the mass ordering,
- CP violation in the lepton sector (Dirac phase δ),
- not sensitive to the nature of neutrinos (neutrinoless double beta decay to probe Majorana nature).

Exercise 2: Neutrino masses In the following we consider the Inverse seesaw mechanism for neutrino mass generation. In this extension, the SM is extended with right-handed neutrinos ν_R and additional sterile states X. The Lagrangian can be generically cast as

$$\mathcal{L}_{\text{ISS}} = -Y^\nu_{ij}\, \overline{L_i}\, \tilde{H}\, \nu_{jR} - M_R^{ij}\, \overline{\nu_{iR}}\, X_j - \frac{1}{2}\, \mu_R\, \overline{\nu_{iR}}\, \nu^C_{jR} - \frac{1}{2}\, \mu_X\, \overline{X^C_i}\, X_j\,, \tag{21}$$

in which M_R is a lepton-number conserving mass matrix, while μ_R and μ_X are symmetric Majorana mass matrices, which break lepton number. Neglecting μ_R and considering only one generation of ν_L and ν_R with two singlet states X_1 and X_2, the ISS mass matrix after EWSB reads, in flavour space $(\nu_L,\, \nu^C_R,\, X_1,\, X_2)$

$$M_{\text{ISS}} = \begin{pmatrix} 0 & m_D & 0 & 0 \\ m_D & 0 & M_R & 0 \\ 0 & M_R & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}, \tag{22}$$

with $m_D = Y^\nu v/\sqrt{2}$ as in the type I seesaw.

Considering the limit $\mu_1, \mu_2 \ll m_D \ll M_R$, the mass matrix can be written as

$$M_{\text{ISS}} = M_0 + \Delta M = \begin{pmatrix} 0 & m_D & 0 & 0 \\ m_D & 0 & M_R & 0 \\ 0 & M_R & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}, \tag{23}$$

and then one can perturbatively diagonalise it.

a) Derive the neutrino masses $m_1^{(0)}, m_2^{(0)}, m_3^{(0)}, m_4^{(0)}$ at 0th order (i.e. the eigenvalues of M_0).

b) To lift the degeneracy between $m_1^{(0)}$ and $m_2^{(0)}$ one has to consider the perturbation ΔM. For this we construct the 2×2 matrix in the degenerate subspace:

$$\delta M_{ij} = \langle x_i^{(0)} | \Delta M | x_j^{(0)} \rangle\,, \qquad \text{with} \qquad i, j = 1, 2 \tag{24}$$

where $x_{i,j}^{(0)}$ are the eigenvectors associated with the diagonalisation of M_0. The eigenvalues $m_1^{(1)}$ and $m_2^{(1)}$ of this perturbation matrix correspond to the correction to the 0th order masses. What is the mass of the active neutrino? What can you conclude about the ISS mass spectrum?

Solution:

a) To diagonalise M_0 and find the mass eigenvalues $m_i^{(0)}$, one has to compute $\det(M_0 - \lambda \not{1}) = 0$. This gives $\lambda^2 - M^2 - m_D^2 = 0$, so the eigenvalues are

$$m_1^{(0)} = m_2^{(0)} = 0\,, \qquad m_3^{(0)} = -\sqrt{M^2 + m_D^2}\,, \qquad m_4^{(0)} = \sqrt{M^2 + m_D^2}\,, \tag{25}$$

since the physical mass has to be positive, we can always absorb the minus sign of $m_3^{(0)}$ by rephasing the matrix diagonalising M_0.

b) The eigenvectors $v_i^{(0)}$ are given by $M_0.v_i^{(0)} = m_i^{(0)}.v_i^{(0)}$ and we have $v_i^{(0)}.v_j^{(0)} = 0$. Denoting $v_i^{(0)} = (\alpha_i, \beta_i, \gamma_i, \delta_i)$ we find the following identities for the first two eigenvectors needed for the computation of the perturbation

$$m_D\beta_i = 0\,, \qquad m_D\alpha_i + M\gamma_i = 0\,, \qquad M\beta_i = 0\,, \qquad i = 1, 2\,. \tag{26}$$

leading to $\alpha_i = -(M/m_D)\gamma_i$, $\beta_i = 0$ and δ_i is unconstrained. Using the orthormonal properties of the eigenvectors, we have

$$v_1^{(0)} = \begin{pmatrix} -\dfrac{M}{\sqrt{M^2 + m_D^2}} \\ 0 \\ \dfrac{m_D}{\sqrt{M^2 + m_D^2}} \\ 0 \end{pmatrix}, \qquad v_2^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{27}$$

The matrix in the degenerate subspace is thus

$$\delta M_{ij} = \langle x_i^{(0)} | \Delta M | x_j^{(0)} \rangle\,, = \begin{pmatrix} \dfrac{m_D^2}{M^2 + m_D^2}\mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}\,. \tag{28}$$

It is then easy to read $m_1^{(1)} = \dfrac{m_D^2}{M^2 + m_D^2}\mu_1$ and $m_2^{(1)} = \mu_2$ giving respectively the active neutrino mass and a "light" sterile mass. Indeed, contrary to the type I seesaw the heavy sterile states do not need to be extremely heavy to give the correct light neutrino masses, as the suppression comes from the scale of lepton number violation (i.e. the matrix μ) that can be low. For instance the correct light neutrino spectrum can be recovered for $M_R \sim$ TeV with $\mathcal{O}(10^{-3})$ Yukawa couplings and $\mu \sim 1$ MeV. Moreover depending on the number of new fields, if $n_X > n_{\nu_R}$, the ISS spectrum features 3 different scales: the light active neutrinos, "light" sterile states and heavy states.