Standard Model and Beyond
Sheet 8

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Exercise 1: Anomalies

A symmetry of a classical theory is generally also a symmetry of the quantum theory with the same Lagrangian. In this exercise, we explore the cases in which quantum effects spoil the conservation condition $\partial_\mu J^\mu = 0$, and the symmetry is said to be anomalous.
Consider two generic abelian currents associated to a vector and an axial-vector symmetries,

$$J^\mu = \overline{\psi}\gamma^\mu\psi\,, \text{ and } J^{\mu\,5} = \overline{\psi}\gamma^\mu\gamma^5\psi\,.$$

The equation of motion implies $\partial_\mu J^\mu = 0$ and $\partial_\mu J^{\mu\,5} = 2im\overline{\psi}\gamma^5\psi$, where m is the fermion mass. In the massless limit, both currents are classically conserved. We want to check if the conservation laws hold in the quantum theory with massless fermions.

a) Consider the correlation function $\langle J^{\alpha\,5}(x)J^\mu(y)J^\nu(z)\rangle$ in momentum space:

$$\begin{aligned}
&iM_5^{\alpha\mu\nu}(p,q_1,q_2)\,(2\pi)^4\delta^4(p-q_1-q_2)\\
&=\int d^4x d^4y d^4z\, e^{-ipx}e^{iq_1y}e^{iq_2z}\left\langle J^{\alpha 5}(x)J^\mu(y)J^\nu(z)\right\rangle\\
&=\int d^4x d^4y d^4z\, e^{-ipx}e^{iq_1y}e^{iq_2z}\left\langle\left[\bar\psi(x)\gamma^\alpha\gamma^5\psi(x)\right]\left[\bar\psi(y)\gamma^\mu\psi(y)\right]\left[\bar\psi(z)\gamma^\nu\psi(z)\right]\right\rangle
\end{aligned}$$

and check explicitly the result of $\partial_\alpha\langle J^{\alpha\,5}(x), J^\mu(y), J^\nu(z)\rangle$ and $\partial_\mu\langle J^{\alpha\,5}(x), J^\mu(y), J^\nu(z)\rangle$ (or, equivalently, of $p_\alpha M_5^{\alpha\mu\nu}$ and $q_{1\mu}M_5^{\alpha\mu\nu}$).
Tip 1: Identify and compute the two 1-loop diagrams contributing to $M_5^{\alpha\mu\nu}$.
Tip 2: Use the most general form for the loop momentum as $k^\mu + b_1 q_1^\mu + b_2 q_2^\mu$ for the first diagram, and, to preserve Bose symmetry, $k^\mu + b_2 q_1^\mu + b_1 q_2^\mu$ for the second diagram.
Tip 3: You might need the following result: linearly divergent integrals that would vanish if we could shift the argument, are actually finite

$$\Delta^\alpha(a^\mu) = \int\frac{d^4k}{(2\pi)^4}\left(F^\alpha[k+a]-F^\alpha[k]\right) = \frac{i}{32\pi^2}a^\mu A\,,$$

where A is defined as

$$\lim_{k_E\to\infty}F^\alpha(k_E) = A\frac{k_E^\alpha}{k_E^4}\,.$$

b) Show that if $b_1 = b_2$ the vector current is broken, while for $b_1 - b_2 = 1$ the vector current is conserved but the axial current is not. The latter choice is the most logical, since the axial symmetry is already broken by the fermion masses. Verify that

$$p_\alpha M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}q_1^\rho q_2^\sigma\,,$$

corresponding, when coupling the current to a $U(1)$ gauge field, to the result

$$\partial_\mu J^{\mu\,5} = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,.$$

c) Generalise the previous equation to the case of non-abelian symmetries. Show that the anomalous term becomes

$$\partial_\mu J^{\mu\,5} = -\frac{e^2}{16\pi^2}\mathrm{Tr}\left[T^a\{T^b\,,T^c\}\right]\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,.$$

where $T^{a,b,c}$ are the generators of the symmetries associated to the axial-vector-vector currents (the representation depending on the fermions running in the loops).

Exercise 2: Cancellation of Gauge Anomalies in the SM

Consider the currents associated to the SM gauge group $SU(3)_{\mathrm{QCD}} \times SU(2)_W \times U(1)_Y$.

a) Check that $\partial_\alpha\langle J_i^\alpha J_j^\mu J_k^\nu\rangle = 0$ for i, j, k any of the SM group, by computing the trace of the generators in each combination.

Exercise 3: Global Anomalies

a) B and L: Important example of a global symmetries of the Standard Model Lagrangian are baryon and lepton number, for which all quarks have $B = 1/3\,, L = 0$ and leptons have $B = 0\,, L = 1$. Check whether these global symmetries are anomalous under the SM gauge groups. Is there a combination of B and L which is non-anomalous?

b) $\pi^0 \to \gamma\gamma$: Considering only the two lightest flavors u and d in the massless limit, QCD has a global vector and axial symmetry $SU(2)_V \times U(1)_V \times SU(2)_A \times U(1)_A$. $SU(2)_A$ is spontaneously broken, giving rise to the pion triplet of Goldstone bosons. Using the fact that $\left\langle \pi^a(q)\left|\partial^\mu J_\mu^a(y)\right|\Omega\right\rangle = q^2 f_\pi e^{i\vec{q}\cdot\vec{y}}$, compute the decay rate for the process $\pi^0 \to \gamma\gamma$, verifying that

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha_{\mathrm{EM}}^2}{64\pi^3}\frac{m_\pi^3}{f_\pi^2}$$