Spring Semester 2024
UZH & ETH
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Standard Model and Beyond
Sheet 7

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https://www.physik.uzh.ch/en/teaching/PHY563/FS2024.html

Issued: 22.04.2024
Due: 25.04.2024

Exercise 1: $\mu \to e\gamma$

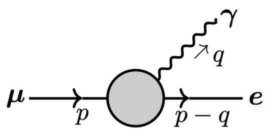
This exercise will be corrected in two tutorials.

The discovery of neutrino oscillations imply that they are massive and that there is lepton mixing. This opens the door to charged lepton flavour violation (cLFV), such as the process $\mu \to e\gamma$ forbidden in the Standard Model. In the following we consider the Standard Model with three massive Dirac neutrinos, such as

$$\nu_\alpha = \sum_i U_{\alpha i}\nu_i \,, \qquad \text{with} \qquad \alpha = e, \mu, \tau; \;\; i = 1, 2, 3 \,. \tag{1}$$

The general diagram for the process $\mu \to e\gamma$ is

$$\mathcal{M}(\mu \longrightarrow e + \gamma) = \epsilon^\lambda \mathcal{M}_\lambda \quad = $$



By Lorentz invariance, the amplitude should be of the form

$$\mathcal{M}_\lambda = \bar{u}_e(p-q)\left[iq^\nu \sigma_{\lambda\nu}(A + B\gamma_5) + \gamma_\lambda(C + D\gamma_5) + q_\lambda(E + F\gamma_5)\right] u_\mu(p) \,, \tag{2}$$

where A, B, C, D, E, F are Lorentz-invariant coefficients (so-called invariant amplitudes).

a) Using the Ward identity show that, in the limit $m_e \to 0$, the amplitude reduces to

$$\epsilon^\lambda \mathcal{M}_\lambda = \epsilon^\lambda \bar{u}_e(p-q) A i q^\nu \sigma_{\lambda\nu}(1 + \gamma_5) u_\mu(p) \,. \tag{3}$$

b) Applying the Gordon decomposition

$$\bar{u}(k)\gamma^\mu \Gamma u(l) = \bar{u}(k)\left[\frac{k^\mu + l^\mu}{2m} + \frac{i\sigma^{\mu\nu}}{2m}(k_\nu - l_\nu)\right]\Gamma u(l) \,, \tag{4}$$

where Γ can be 1 or γ_5, show that

$$\epsilon^\lambda \mathcal{M}_\lambda = \bar{u}_e(p-q) 2A(1 + \gamma_5)\left[(p \cdot \epsilon) - m_\mu(\gamma \cdot \epsilon)\right] u_\mu(p) \,. \tag{5}$$

Thus in what follows we can focus only on the terms containing $(p \cdot \epsilon)$ to compute A.

The rare process $\mu \to e\gamma$ arises from one-loop diagrams involving a massive neutrino ν_i in the loop. Working in Feynman gauge, the different Feynman diagrams contributing to the decay are given by

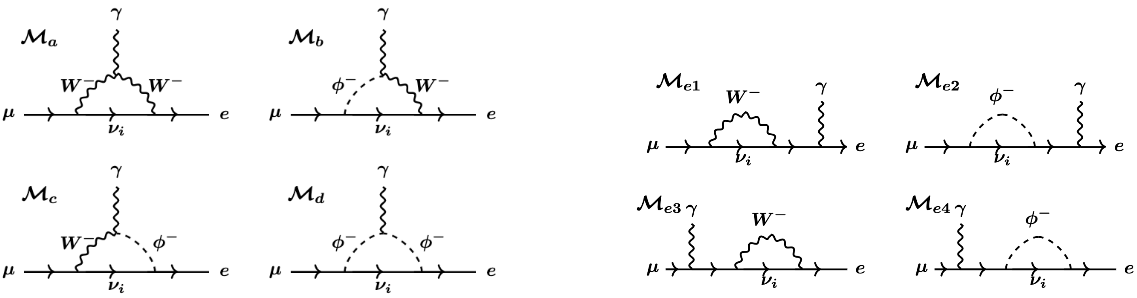


Figure 1: Feynman diagrams contributing to $\mu \to e\gamma$, with ϕ the Goldstone boson associated to W.

c) From the vertices present in the loop diagrams $e1, e2, e3$ and $e4$ explain why we can ignore these four diagrams in the computation of the amplitude given in Eq. 5.

Focusing on the diagrams a, b, c, d, we will consider the momentum assignments depicted in Fig. 2 in what follows.

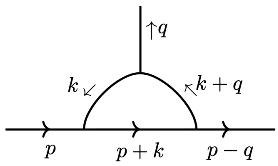
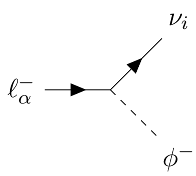
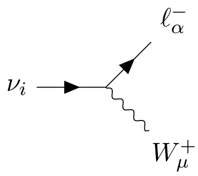


Figure 2: Momentum assignments.

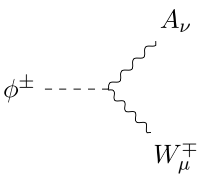
Our goal is now to compute the contribution from diagram b, and extract the invariant amplitude A_b. To do so, the relevant Feynman rules are given by



$$= i\frac{g}{\sqrt{2}M_W}\,\mathcal{U}_{\alpha i}\,[m_i\,P_L - m_\alpha P_R]$$



$$= -i\frac{g}{\sqrt{2}}\,\mathcal{U}^*_{\alpha i}\,\gamma_\mu\,P_L$$



$$= ieM_W\,g_{\mu\nu}$$

Table 1: Feynman rules for W interactions and associated Goldstone bosons in the SM with massive Dirac neutrinos.

d) Write down the amplitude $\mathcal{M}_b$ and show that it can be written as

$$\mathcal{M}_b = \frac{ieg^2}{2}\int \frac{d^4k}{(2\pi)^4}\sum_i \frac{U^*_{ei}U_{\mu i}}{(p+k)^2-m_i^2}\frac{1}{(k+q)^2-M_W^2}\frac{1}{k^2-M_W^2}$$
$$\times \bar{u}_e(p-q)\gamma^\lambda P_L[-m_\mu(m_\mu+\not{k})+m_i^2]u_\mu(p)\epsilon_\lambda\,, \quad (6)$$

e) In the above expression, we have to sum over the neutrino mass eigenstates, hence we want to have an expression that will not depend on m_i on the denominator. Expand the contribution in the high-momentum limit and show that the amplitude can be cast as

$$\mathcal{M}_b = i\mathcal{C}\int \frac{d^4k}{(2\pi)^4}\frac{N^\lambda}{[(p+k)^2]^2}\frac{1}{(k+q)^2-M_W^2}\frac{1}{k^2-M_W^2}\epsilon_\lambda\,, \quad (7)$$

with

$$\mathcal{C} = \frac{eg^2}{2}\sum_i U^*_{ei}U_{\mu i}m_i^2 \quad \text{and} \quad N^\lambda = \bar{u}_e(p-q)P_R\gamma^\lambda[-m_\mu(m_\mu+\not{k})+(p+k)^2]u_\mu(p)\,. \quad (8)$$

Before performing the loop integration we introduce the Feynman parameters to combine the propagators in the denominator

$$\frac{1}{D_1^{m_1}D_2^{m_2}...D_n^{m_n}} = \int_0^1 d\alpha_1...d\alpha_n\delta(\sum x_i - 1)\frac{\Pi\alpha_i^{m_i-1}}{[\sum\alpha_i D_i]^{\sum m_i}}\frac{\Gamma(m_1+...+m_n)}{\Gamma(m_1)...\Gamma(m_n)}\,, \quad (9)$$

with $\Gamma(n) = (n-1)!$

f) Show that the denominator can be written as

$$D = [l^2 - (1-\alpha_1)M_W^2]^4 \qquad \text{with} \qquad l = k + \alpha_1 p + \alpha_2 q\,, \quad (10)$$

g) Using the shift of variable $k = l - \alpha_1 p - \alpha_2 q$ and keeping only terms proportional to $(p\cdot\epsilon)$ show that the numerator is given by

$$N^\lambda\epsilon_\lambda = -m_\mu\alpha_2(p\cdot\epsilon)\bar{u}_e(p-q)P_R u_\mu(p)\,, \quad (11)$$

We can now perform the loop integration using

$$\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}\,,$$
$$\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-1}\,. \quad (12)$$

And then perform the integration over the Feynman parameters using

$$\int d\alpha_1 d\alpha_2\frac{\alpha_1\alpha_2}{(1-\alpha_1)^2} = \frac{1}{4}\,. \quad (13)$$

h) What is $\mathcal{M}_b$? Deduce the expression of the invariant amplitude A_b.

We now want to compute the invariant amplitude A_c arising from diagram c.

i) Write down the amplitude $\mathcal{M}_c$. Using the shift of variable $l = k + \alpha_1 p + \alpha_2 q$ and focusing only on the denominator, show that $A_c = 0$ in Feynman gauge (we are only interested in the part proportional to $(p \cdot \epsilon)$).

The invariant amplitude coming from diagram d turns out to be $A_a = -A_d$. Using this result, we can finally evaluate the decay rate.

j) Given that $\Gamma(\mu \to e\nu\bar{\nu}) = m_\mu^5 G_F^2/(192\pi^3)$ compute the branching ratio $\mathrm{BR}(\mu \to e\gamma) = \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to e\nu\bar{\nu})}$. What can you conclude?

Solution:

a) The first step is to replace the polarisation vector ϵ^λ by the momentum q^λ of the photon. The Ward identity gives

$$\begin{aligned} q^\lambda \mathcal{M}_\lambda = 0 &= \bar{u}_e(p-q)\left[iq^\lambda q^\nu \sigma_{\lambda\nu}(A+B\gamma_5) + q^\lambda\gamma_\lambda(C+D\gamma_5) + q^\lambda q_\lambda(E+F\gamma_5)\right] u_\mu(p) \\ &= \bar{u}_e(p-q)\left[iq^\lambda q^\nu \sigma_{\lambda\nu}(A+B\gamma_5) + \not{q}(C+D\gamma_5) + q^2(E+F\gamma_5)\right] u_\mu(p)\,, \end{aligned} \tag{14}$$

The first term is proportional to $q^\lambda q^\nu \sigma_{\lambda\nu}$ and hence vanishes, while the last one is zero since the photon is on-shell ($q^2 = 0$). So we just have to work out the second term:

$$\begin{aligned} 0 &= \bar{u}_e(p-q)\left[\not{q}(C+D\gamma_5)\right] u_\mu(p) \\ &= \bar{u}_e(p-q)\left[(\not{q}+\not{p}-\not{p})(C+D\gamma_5)\right] u_\mu(p)\,. \end{aligned} \tag{15}$$

Using $\not{p}u_\mu(p) = m_\mu u_\mu(p)$ and $\bar{u}_e(p-q)(\not{p}-\not{q}) = \bar{u}_e(p-q)m_e$, we have

$$0 = \bar{u}_e(p-q)\left[(m_\mu - m_e)(C+D\gamma_5)\right] u_\mu(p)\,, \tag{16}$$

and since $m_\mu \neq m_e$ we arrive at $C = D = 0$.

Going back to the matrix element multiplied by the polarisation vector we have

$$\epsilon^\lambda \mathcal{M}_\lambda = \bar{u}_e(p-q)\left[i\epsilon^\lambda q^\nu \sigma_{\lambda\nu}(A+B\gamma_5) + \epsilon^\lambda q_\lambda(E+F\gamma_5)\right] u_\mu(p)\,, \tag{17}$$

and by construction of the polarisation vector $(\epsilon \cdot q) = 0$. Finally since we are interested in the limit $m_e = 0$, the outgoing electron will be left-handed, and the amplitude will be of the form $P_L u_e$ or in our case $\bar{u}_e P_R$. In this approximation we have $A = B$ and recover

$$\epsilon^\lambda \mathcal{M}_\lambda = \epsilon^\lambda \bar{u}_e(p-q) A i q^\nu \sigma_{\lambda\nu}(1+\gamma_5) u_\mu(p)\,. \tag{18}$$

b) From the Gordon identity we have

$$\begin{aligned} \bar{u}(p-q)\gamma^\mu \Gamma u(p) &= \bar{u}(p-q)\left[\frac{(p-q)^\mu + p^\mu}{2m_\mu} + \frac{i\sigma^{\mu\nu}}{2m_\mu}((p-q)_\nu - p_\nu)\right]\Gamma u(p) \\ &= \bar{u}(p-q)\frac{1}{2m_\mu}\left[(2p-q)^\mu - i\sigma^{\mu\nu}q_\nu\right]\Gamma u(p)\,, \end{aligned} \tag{19}$$

leading to

$$\bar{u}(p-q)i\sigma^{\mu\nu}q_\nu(1+\gamma_5)u(p) = \bar{u}(p-q)\left[(2p-q)^\mu - 2m_\mu\gamma^\mu\right](1+\gamma_5)u(p)\,, \tag{20}$$

Plugging this expression into Eq. (3)

$$\begin{aligned} \epsilon^\lambda \mathcal{M}_\lambda &= \epsilon^\lambda \bar{u}_e(p-q)A\left[(2p-q)_\lambda - 2m_\mu\gamma_\lambda\right](1+\gamma_5)u_\mu(p) \\ &= \bar{u}_e(p-q)2A\left[(p\cdot\epsilon) - m_\mu(\gamma\cdot\epsilon)\right](1+\gamma_5)u_\mu(p)\,. \end{aligned} \tag{21}$$

Thus in order to compute the decay rate $\mu \to e\gamma$ we can only focus on the computation of the terms involving $(p \cdot \epsilon)$ to extract the invariant amplitude A.

c) Looking at the vertices of diagrams e_1 and e_3 we know that the amplitude will be proportional to

$$\mathcal{M}_{e_1}, \mathcal{M}_{e_3} \;\propto\; \bar{u}_e(\gamma_\alpha\gamma_\mu\gamma_\nu g^{\mu\nu})u_\mu\epsilon^\alpha \propto \bar{u}_e(\gamma_\alpha)u_\mu\epsilon^\alpha \propto \bar{u}_e(\gamma\cdot\epsilon)u_\mu \tag{22}$$

hence we cannot have a term proportional to $(p \cdot \epsilon)$, and we can then neglect these diagrams in the determination of the invariant amplitude A. A similar argument holds for diagrams e_2 and e_4.

We now proceed to compute the contribution from diagram b

d) From the Feynman rules and the momentum assignments, the amplitude reads

$$\begin{aligned}
\mathcal{M}_b &= -i\sum_i \int \frac{d^4k}{(2\pi)^4}\, \bar{u}_e(p-q)\left(-i\frac{g}{\sqrt{2}}U^*_{ei}\gamma_\alpha P_L\right)\frac{i(\not{p}+\not{k}+m_i)}{(p+k)^2-m_i^2}\left(i\frac{g}{\sqrt{2}M_W}U_{\mu i}(m_iP_L-m_\mu P_R)\right)u_\mu(p) \\
&\quad \times\frac{-ig^{\alpha\beta}}{(k+q)^2-M_W^2}\frac{i}{k^2-M_W^2}(ieM_W g_{\lambda\beta})\epsilon^\lambda \qquad (23)\\
&= i\frac{eg^2}{2}\sum_i\int\frac{d^4k}{(2\pi)^4}\,\bar{u}_e(p-q)\,(U^*_{ei}\gamma_\lambda P_L)\,\frac{\not{p}+\not{k}+m_i}{(p+k)^2-m_i^2}\,(U_{\mu i}(m_iP_L-m_\mu P_R))\,u_\mu(p) \\
&\quad \times\frac{1}{(k+q)^2-M_W^2}\frac{1}{k^2-M_W^2}\epsilon^\lambda \qquad (24)
\end{aligned}$$

Focusing on the numerator we have

$$\gamma_\lambda P_L(\not{p}+\not{k}+m_i)(m_iP_L-m_\mu P_R) = \gamma_\lambda m_i^2 P_L - \gamma_\lambda(\not{p}+\not{k})m_\mu P_R \qquad (25)$$

using

$$(\not{p}+\not{k})P_R u_\mu(p) = P_L(\not{p}+\not{k})u_\mu(p) = P_L(m_\mu+\not{k})u_\mu(p) \qquad (26)$$

we arrive at

$$\begin{aligned}
\mathcal{M}_b &= \frac{ieg^2}{2}\int\frac{d^4k}{(2\pi)^4}\sum_i\frac{U^*_{ei}U_{\mu i}}{(p+k)^2-m_i^2}\frac{1}{(k+q)^2-M_W^2}\frac{1}{k^2-M_W^2} \\
&\quad \times\bar{u}_e(p-q)\gamma^\lambda P_L[-m_\mu(m_\mu+\not{k})+m_i^2]u_\mu(p)\epsilon_\lambda\,, \qquad (27)
\end{aligned}$$

e) Having to sum over the neutrino mass eigenstates, we want to have an expression that will not depend on m_i in the denominator. For this, we expand the contribution in the high momentum limit:

$$\sum_i\frac{U^*_{ei}U_{\mu i}}{(p+k)^2-m_i^2} = \sum_i U^*_{ei}U_{\mu i}\left[\frac{1}{(p+k)^2}+\frac{m_i^2}{[(p+k)^2]^2}+...\right] \simeq \sum_i U^*_{ei}U_{\mu i}\frac{m_i^2}{[(p+k)^2]^2}\,, \qquad (28)$$

where the leading term vanishes due to the GIM mechanism. Similarly we have

$$\sum_i\frac{U^*_{ei}U_{\mu i}m_i^2}{(p+k)^2-m_i^2} \simeq \sum_i U^*_{ei}U_{\mu i}\frac{m_i^2}{(p+k)^2}\,. \qquad (29)$$

The amplitude can then be cast as

$$\begin{aligned}
\mathcal{M}_b &= \frac{ieg^2}{2}\sum_i U^*_{ei}U_{\mu i}m_i^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(p+k)^2]^2}\frac{1}{(k+q)^2-M_W^2}\frac{1}{k^2-M_W^2} \\
&\quad \times\bar{u}_e(p-q)P_R\gamma^\lambda[-m_\mu(m_\mu+\not{k})+(p+k)^2]u_\mu(p)\epsilon_\lambda \\
&= i\mathcal{C}\int\frac{d^4k}{(2\pi)^4}\frac{N^\lambda}{[(p+k)^2]^2}\frac{1}{(k+q)^2-M_W^2}\frac{1}{k^2-M_W^2}\epsilon_\lambda\,, \qquad (30)
\end{aligned}$$

with

$$\mathcal{C} = \frac{eg^2}{2}\sum_i U^*_{ei}U_{\mu i}m_i^2 \quad \text{and} \quad N^\lambda = \bar{u}_e(p-q)P_R\gamma^\lambda[-m_\mu(m_\mu+\not{k})+(p+k)^2]u_\mu(p)\,, \qquad (31)$$