NMR → MRI
(spin precession of nuclei)

from PHY 117:
angular momentum
torque
magnetic moment

PHY127 FS2024

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Linear motion

momentum $\bar{p} = m\bar{v}$

$$\frac{d\bar{p}}{dt} = m\frac{d\bar{v}}{dt} = m\bar{a} \quad \text{(mass constant)}$$

Newton's 2^{nd} law:

$$\Sigma \bar{F} = m\bar{a} = \frac{d\bar{p}}{dt}$$

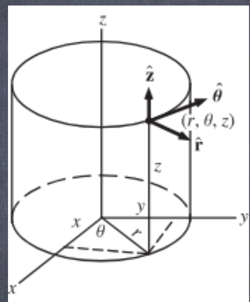
Angular motion

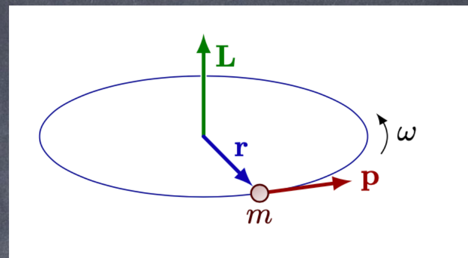
angular momentum $\bar{L} = \bar{r} \times \bar{p} = \bar{r} \times m\bar{v}$

Newton's 2^{nd} law for rotation:

$$\underset{\text{torque}}{\Sigma \bar{\tau}} = \frac{d\bar{L}}{dt} \qquad \bar{\tau} = \bar{r} \times \bar{F}$$

cylindrical coordinates





(a) Uniform circular motion.

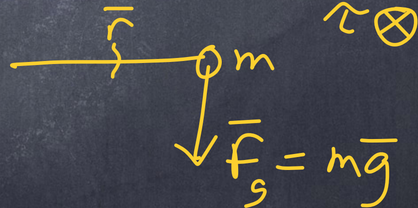
$\omega = \frac{v}{r}$

(a) when $\bar{r} \perp \bar{v}$, then

$$\bar{L} = \bar{r} \times m\bar{v} = rmv = rm\omega r$$

① $\boxed{L = mr^2\omega}$

side view



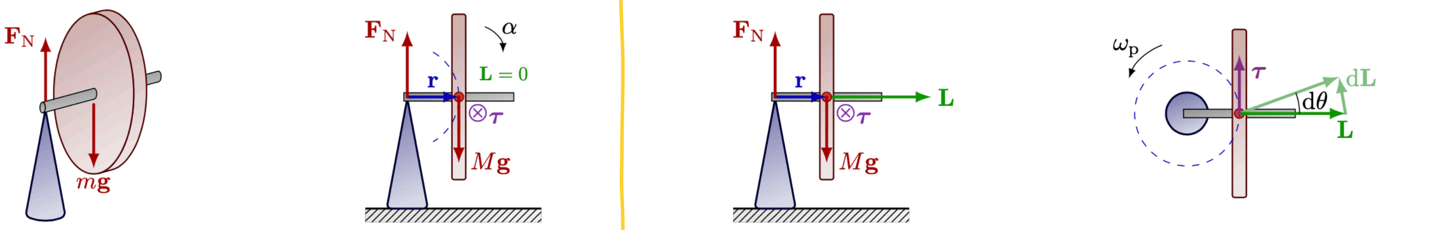
② $\boxed{\frac{d\bar{L}}{dt} = \bar{\tau}}$

$$d\bar{L} = \bar{\tau}\, dt = (\bar{r} \times \bar{F})\, dt$$

If force is due to gravity, then $\bar{F}_g = m\bar{g}$ & $\bar{r} \perp \bar{F}$,

so $d\bar{L} = (\bar{r} \times m\bar{g})\, dt = rmg\, dt\,(\hat{\theta})$

(from above)



(a) The handle allows the disk to spin around its axis and around the pivot. (b) The disk does not spin, and it will fall down due to an unbalanced torque τ. (c) The disk spins, creating an angular momentum $\mathbf{L}$. Torque τ will cause a precession. (d) Torque τ perpendicular to angular momentum $\mathbf{L}$, will only change its direction.

θ changing

(a+b) forces balance, torque $\bar{\tau} = \bar{r} \times M\bar{g} = rmg\hat{\theta}$ causes wheel to rotate (fall) when $\bar{L} = 0$

(c) wheel spins with angular speed $\omega = \frac{v}{r}$, we have $L = mr^2\omega$ (from 1)

(d) torque from $\bar{r} \times \bar{F}$, causing $\bar{L}$ to change direction

(from 2) $\frac{d\bar{L}}{dt} = \bar{\tau}$ $\quad \boxed{d\bar{L}} = \bar{\tau}\, dt = \boxed{rmg\, dt\, (\hat{\theta})}$ 3

Looking at (d), $dL = L\, d\theta$, $\boxed{d\theta = \frac{dL}{L}}$ 4

3 → 4, $d\theta = \frac{rmg\, dt}{L}$, so $\frac{d\theta}{dt} = \frac{rmg}{L}$, we define $\frac{d\theta}{dt} = \omega_p$ (precession)

5 $\boxed{\omega_p = \frac{rmg}{L}}$ put 1 → 5, $\omega_p = \frac{\cancel{r}\cancel{m}g}{\cancel{m}r^2\omega} = \frac{g}{r\omega}$

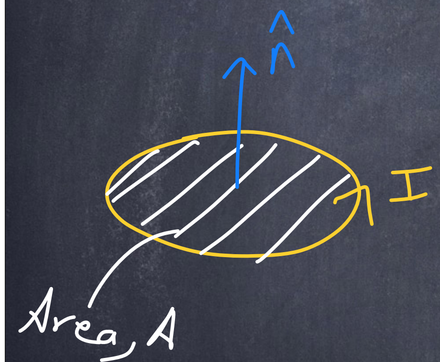
precession $\quad$ spinning wheel

magnetic moment

① Rotating particle carries angular momentum

$$\bar{L} = \bar{r} \times m\bar{v}$$

② A electric current generates a magnetic field.
A current-carrying loop will generate a magnetic moment, $\bar{\mu}$.



$\hat{n}$: normal vector perpendicular to the loop.

$$\bar{\mu} = I A \hat{n}$$

see script 2, § 11.1

③ A rotating charged particle has a magnetic moment related to the angular momentum.

$$\bar{\mu} = g \frac{q}{2m} \bar{L}$$

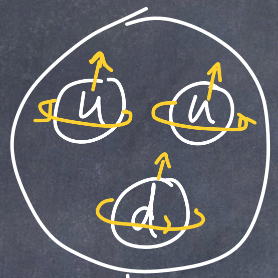
q: electric charge] of particle
m: mass]
g: strength parameter
$g = 1$ if charge + mass have the same distribution

In QM, spin $= \frac{1}{2}$ particle, $g = 2$

④ protons & neutrons are composed of quarks.

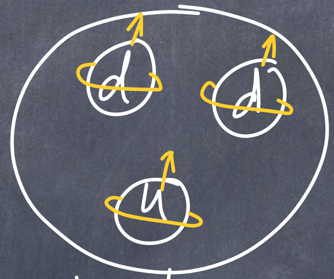
$q(u) = +\frac{2}{3}e$

$q(d) = -\frac{1}{3}e$



proton

$q(p) = +1e$



neutron

$q(n) = 0$

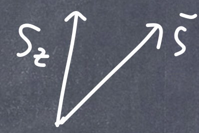
quarks are spin $= \frac{1}{2}$ particles

The spin of a proton & neutron gives it angular momentum. Both have magnetic moments.

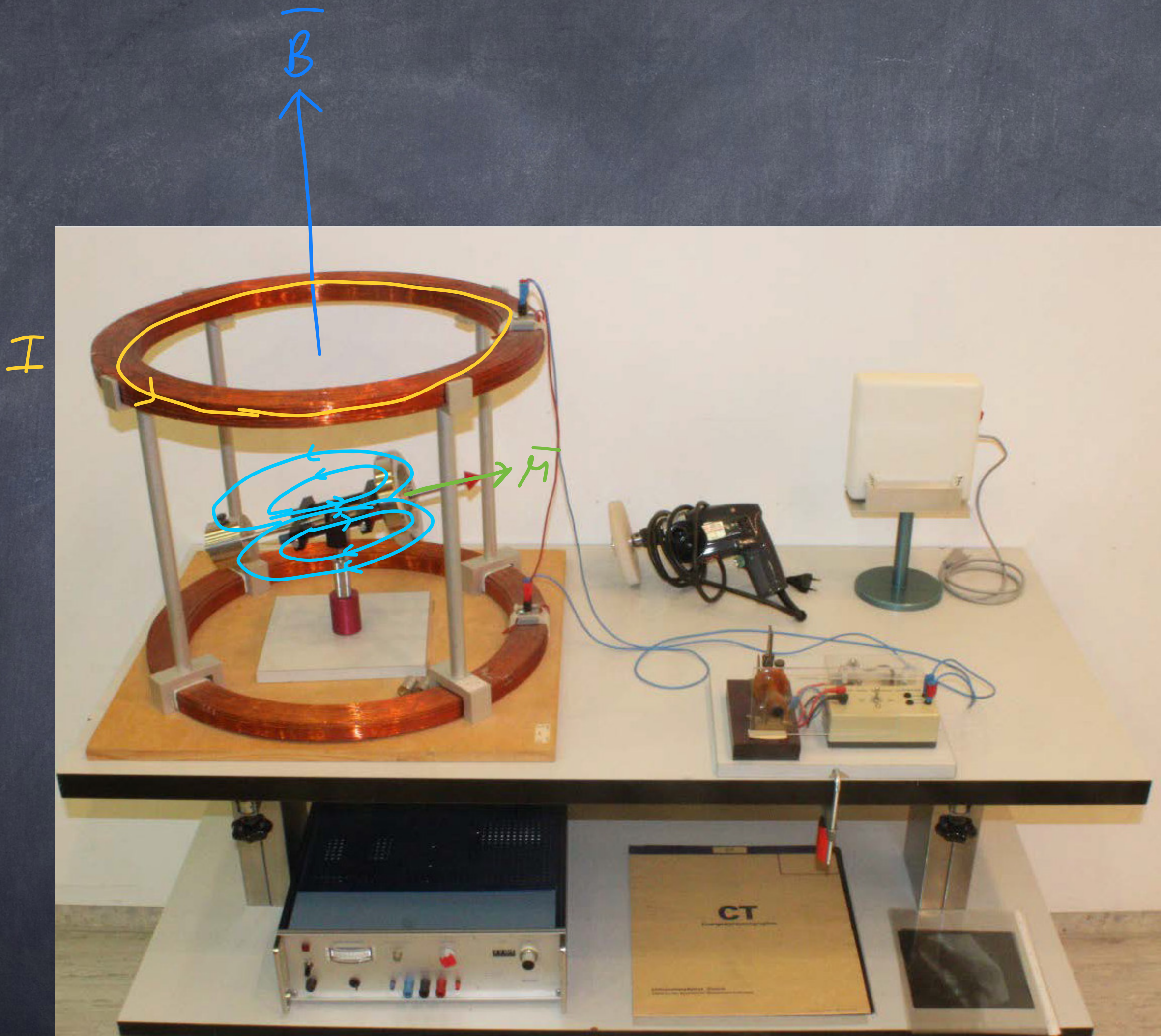
the angular momentum S_z quantized by an integer number of $\frac{1}{2}\hbar = \frac{h}{4\pi}$ (h: Planck's constant)

$\hbar = \frac{h}{2\pi}$

nucleus	spin	S_z	S
proton	$\frac{1}{2}$	$\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
neutron	$\frac{1}{2}$	$\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
deuteron (^2H)	1	$\hbar$	$\sqrt{2}\hbar$
Helium (He)	0	0	0
^{12}C	0	0	0
^{13}C	$1/2$	$1/2\,\hbar$	$\frac{\sqrt{3}}{2}\hbar$
^{14}N	1	$\hbar$	$\sqrt{2}\hbar$
^{16}O	0	0	0
^{19}F	$1/2$	$\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
^{31}P	$1/2$	$1/2\,\hbar$	$\frac{\sqrt{3}}{2}\hbar$



S_z is z-component of $\vec{S}$


B
I
M
CT

The atomic nuclei have nuclear magnetic moments that depend on their nuclear spin.

$$\boxed{\bar{\mu} = g\frac{q}{2m}\bar{L} = \gamma\bar{S}} \quad (8)$$

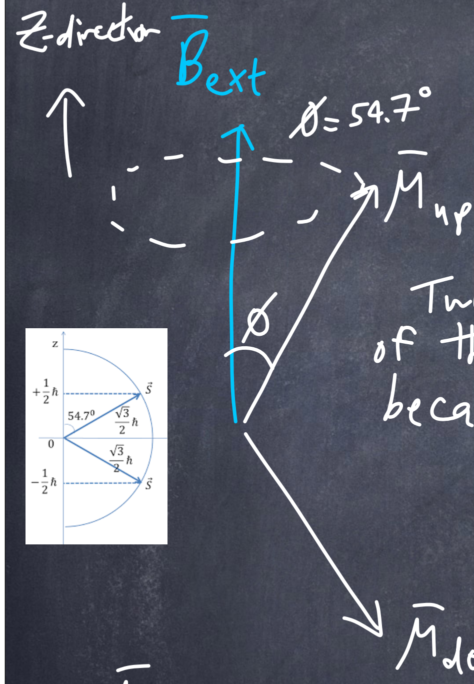
γ : gyromagnetic ratio

$$\gamma = \frac{gq}{2m}$$

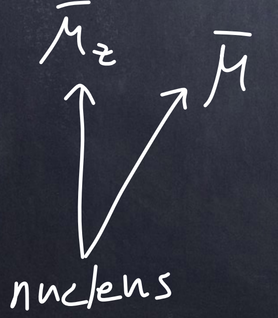
S : spin (angular momentum)

$\bar{\mu} = \gamma\bar{S}$

$\mu_z = \gamma S_z$



Two possible orientations of the magnetic moment, because $\bar{\mu}$ is quantized.



When a magnetic moment and an external magnetic field are not parallel, there is a torque on the magnetic moment.

$$\bar{\tau} = \bar{\mu} \times \bar{B} = \mu B \sin\phi \; (\hat{\tau})$$

Z-component is quantized & points up or down.

$$\bar{\tau} = \mu B \sin\phi = \frac{d\bar{L}}{dt}$$

$$dL = \tau\, dt$$

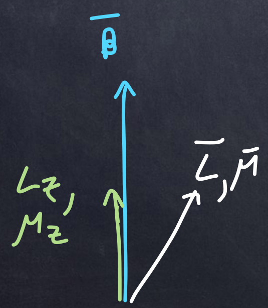
the torque is $\bar{\mu} \times \bar{B}$

$$d\bar{L} = (\bar{\mu} \times \bar{B})\, dt\, (\hat{\tau})$$

$$d\theta = \frac{dL}{L} = \frac{\bar{\mu} \times \bar{B}}{L}\, dt$$

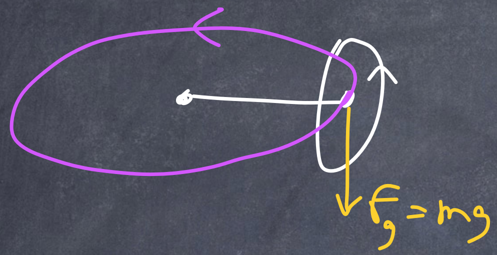
$$w_p = \frac{d\theta}{dt} = \frac{dL}{L} = \frac{\bar{\mu} \times \bar{B}}{L}\, dt$$

$$w_p = \frac{d\theta}{dt} = \frac{\mu B \sin\phi}{L_z} = \frac{\mu B \cancel{\sin\phi}}{L \cancel{\sin\phi}}$$



$$w_p = \frac{\mu B}{L}$$

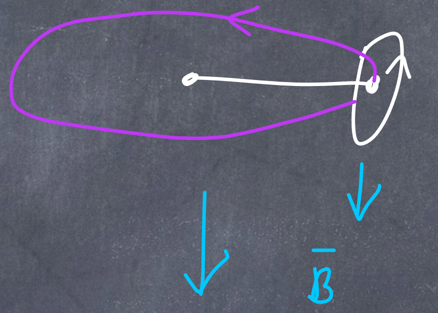
spinning top in gravity



$$\omega_p = \frac{rmg}{L} = \frac{\tau}{L}$$

$\tau = rmg$

spinning magnet in a magnetic field



$$\omega_p = \frac{\mu B}{L} = \frac{\tau}{L}$$

$\tau = \mu B$

Larmor frequency

$$\omega = -\gamma B$$

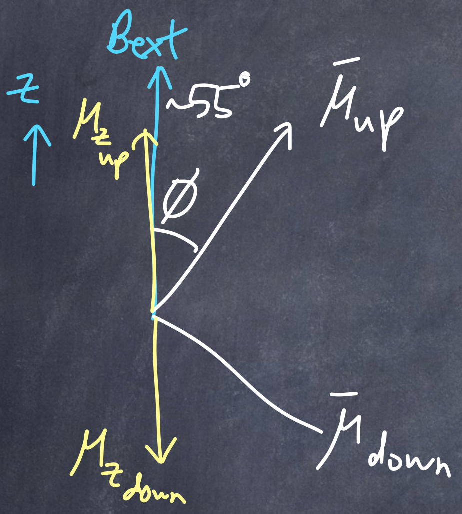
B: external magnetic field

$$\gamma = \frac{-eg}{2m}$$ ← strength factor

gyromagnetic ratio for a particle of charge $-e$

(γ is (+))

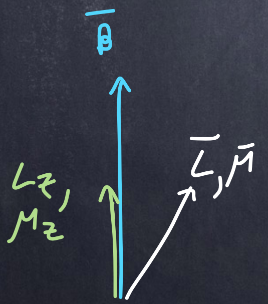
The fact that μ is not aligned with $\bar{B}_{ext}$ gives the nucleus a potential energy, U.

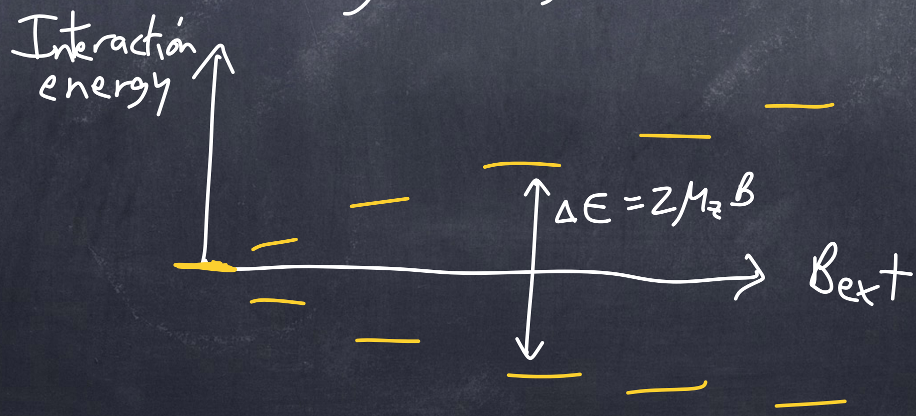


$$U = -\bar{\mu}\cdot\bar{B} = -\mu B\cos\phi = -B(\mu\cos\phi) = -B\mu_z$$

The energy difference between the up + the down state ($\bar{\mu}_{up} + \bar{\mu}_{down}$) is $\Delta E = \mu_z B - (-\mu_z B) = 2\mu_z B$

The energy difference increases with increasing magnetic field B_{ext}.

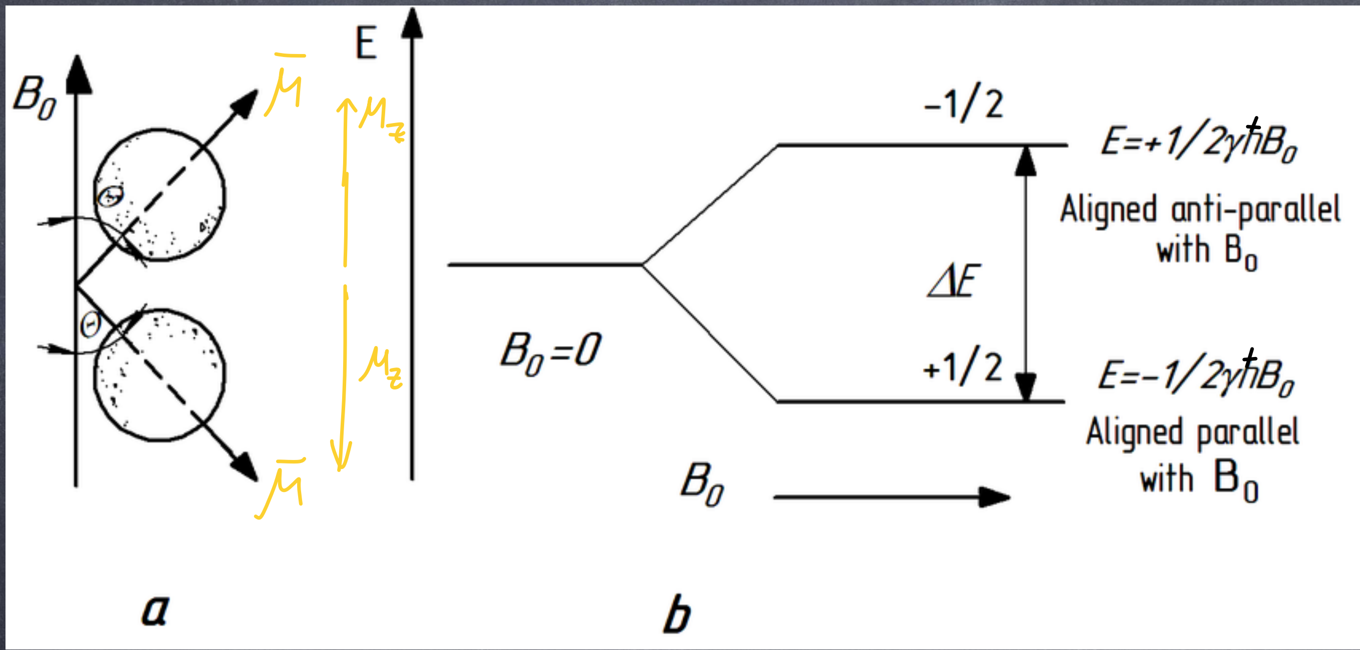




$$\Delta E = 2\mu_z B$$

from ⑧ $\mu_z = \gamma S_z$ $\qquad S_z = \pm\frac{1}{2}\hbar$

so $\Delta E = 2 \cdot \gamma \cdot \frac{1}{2} \cdot B = \gamma\hbar B$



Some numbers:

For $B = 1T$ & a nucleus of hydrogen (1 proton) with nuclear spin of $\frac{1}{2}\hbar$

we would get $\Delta E \sim 2E-7$ eV

we can compare this to the thermal energy of a proton (Hydrogen) at room temperature:

$\sim k_B T \cong 2.5E-2$ eV

The magnetic potential energy is small compared to the thermal energy.

According to the Boltzmann factor for the ratio of the number of spin-up atoms (nuclei) to spin-down atoms (nuclei)

$$\frac{n_{up}}{n_{down}} = e^{-\frac{\Delta E}{k_B T}} = e^{-\frac{2E-7}{2.5E-2}} = 0.999992$$

$\rightarrow$ diff. between n_{up} & n_{down} is only a few parts per million

NMR (nuclear magnetic resonance)

involves adding electromagnetic radiation in units of photon energy, $\epsilon = h\nu$, and then measuring the net (total) absorption of the photons.

$$\Delta\epsilon = 2\mu_z B = h\nu$$

we need very low-frequency photons ~ radio-frequency (MHz)

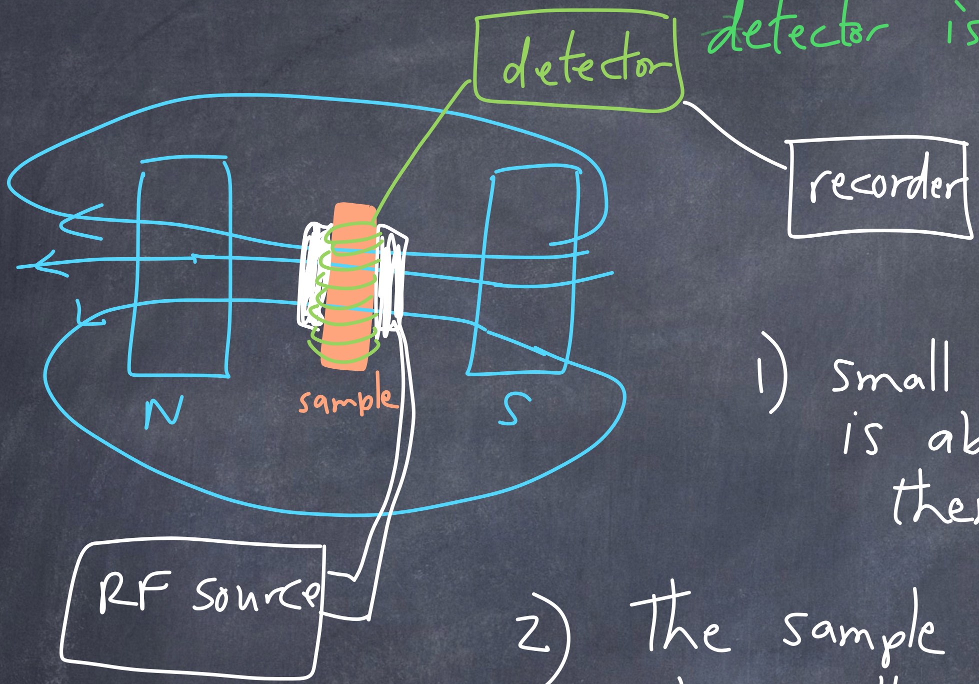
Looking at formula, we see that by either varying ν and fixing $\bar{B}$, or fixing ν and varying $\bar{B}$, we can generate a resonance condition where there will be a net absorption of photon energy causing the nuclei to flip spins to high energy state.

In NMR, RF (radio frequency) is fixed and $\bar{B}$ is varied by small amounts while scanning through for resonance conditions. (see video on wikipedia page)

Einstein showed that the same RF photons that can be absorbed, flipping spins to a higher energy state, can with equal probability, flip the nuclear spin to a lower energy state, emitting a second RF photon with energy ΔE.

If n_{up} + n_{down} were equal, there would be no net absorption. But, since there are slightly more nuclei in the n_{down} state than the n_{up} state, there is a slight net absorption of photons. This is our signal for NMR.

How to detect NMR radiation.



detector is a solenoid that can detect electrical current.

1) small amount of RF radiation is absorbed by our sample, then we turn off the RF.

2) The sample returns to equilibrium, by emitting RF energy.

The net magnetic moment of the sample changes, and this can be detected in the solenoidal coil. (This comes from PHY 117 (script 2)) The changing magnetism of the sample induces a electric current in the solenoid.) (Faraday's Law)

Next time: how to use NMR to determine molecular structure