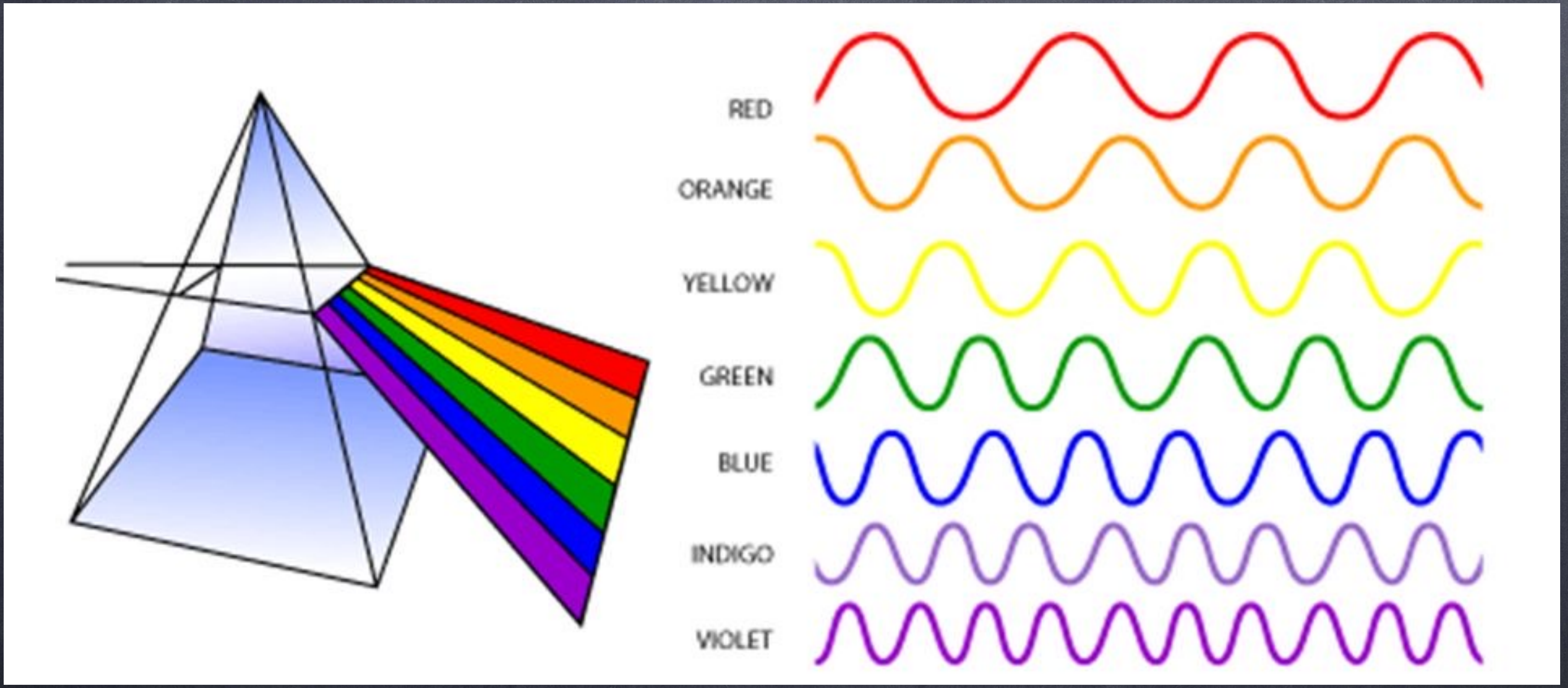
PHY127 FS2024

Prof. Ben Kilminster

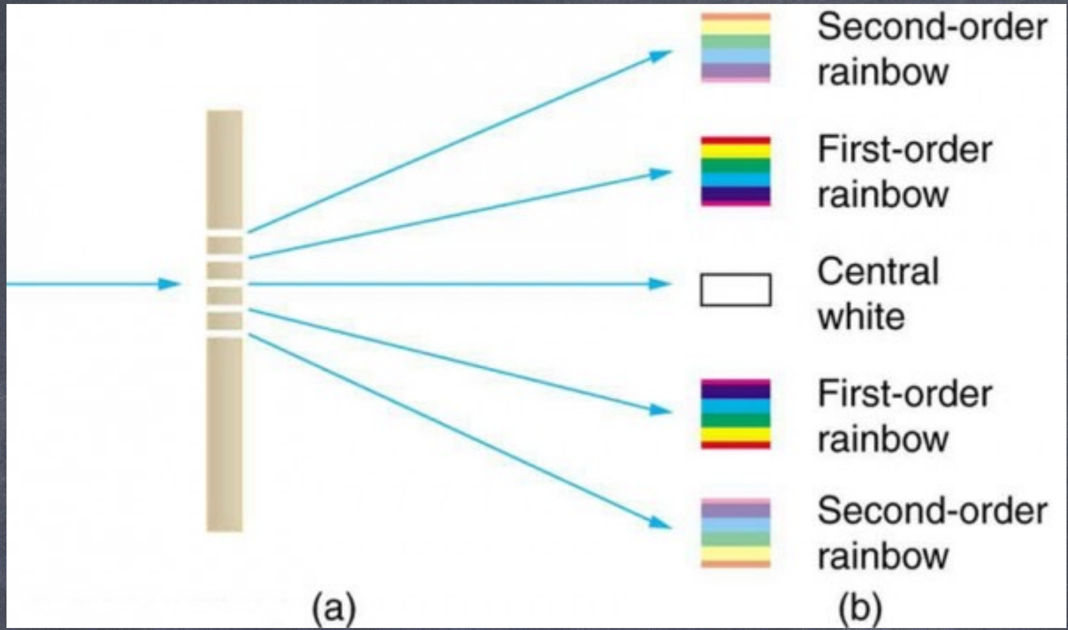
Lecture 6

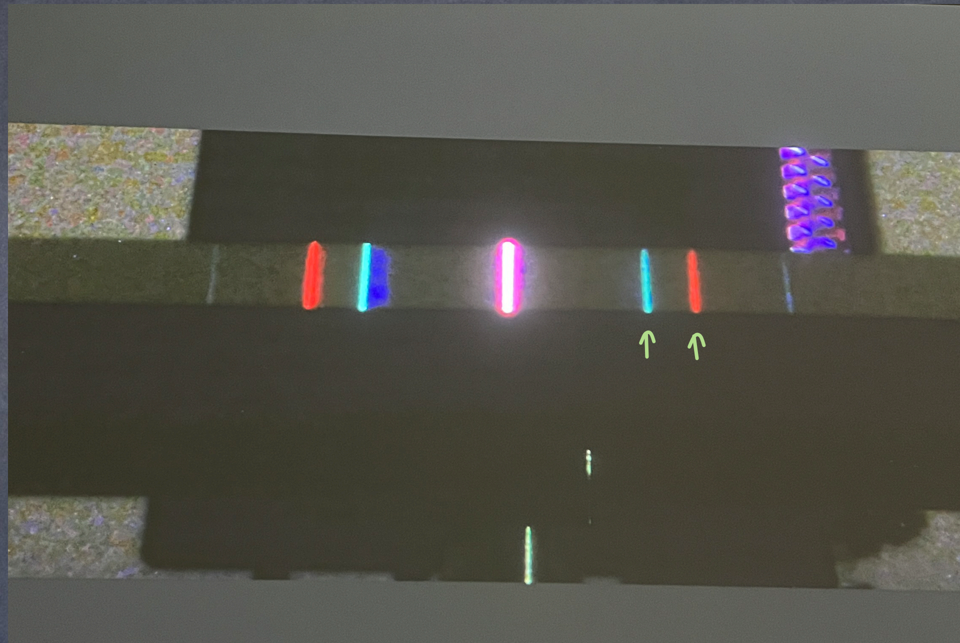
April 12th, 2024

we observe that white light generated from a blackbody can be split into a spectrum of frequencies.

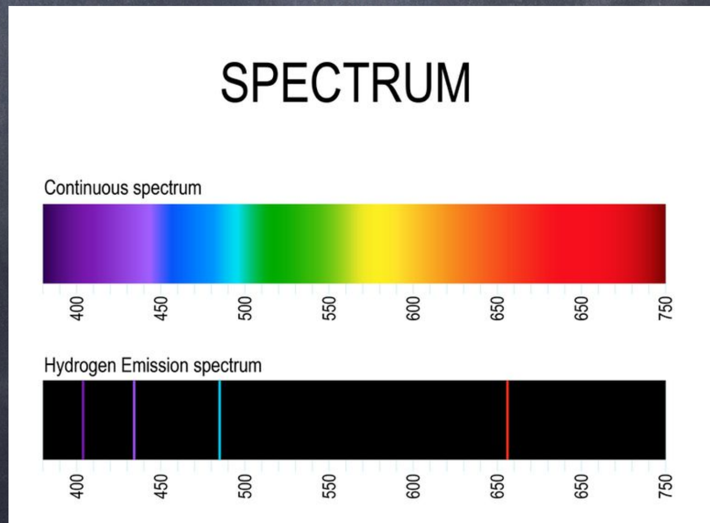


diffraction grating splits light into wavelengths





using a diffraction grating,
we split light from
hydrogen gas



we observe that
hydrogen emits light
of specific wavelengths.

Make a model of the light coming from an atom.

Empirically, light (visible) from hydrogen atom,

Balmer (1885) Balmer series: $\lambda = 364.6 \text{ nm} \left(\frac{m^2}{m^2-4} \right)$ for $m = 3, 4, 5, \ldots$

Extended to other atoms

Rydberg (1888)

$$\boxed{\frac{1}{\lambda} = R \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right)}$$

n_1, n_2: integers

$n_1 > n_2$

R: Rydberg constant

$R = 10.97373 \ \mu m^{-1}$

After advent of Einstein,

$E = \frac{hc}{\lambda} = h\nu$, and quantum mechanics,

Bohr (1913) realized these described energy levels ($E = h\nu$),

general: $$\boxed{\frac{\nu}{c} = \frac{1}{\lambda} = R Z^2 \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right)}$$

n_1, n_2: integers

$n_1 > n_2$

Z: charge of atom

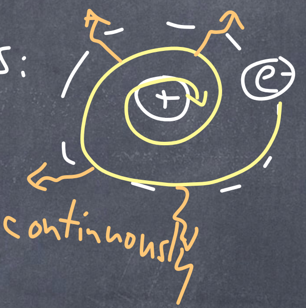
Bohr: hypothesis that violates classical physics.

There are allowed transitions in energy such that

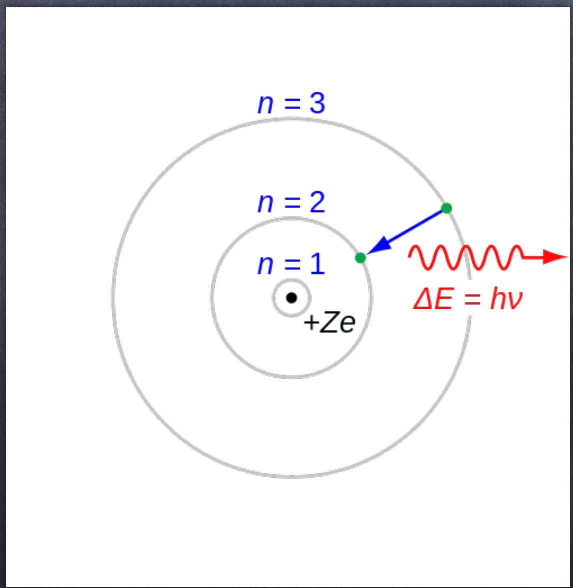
$$\boxed{\nu = \frac{\epsilon_i - \epsilon_f}{h}} \quad (1)$$

ϵ_i, ϵ_f: initial + final energies.

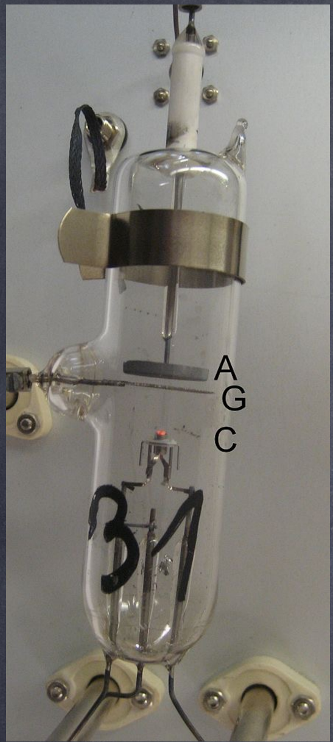
Classical physics:

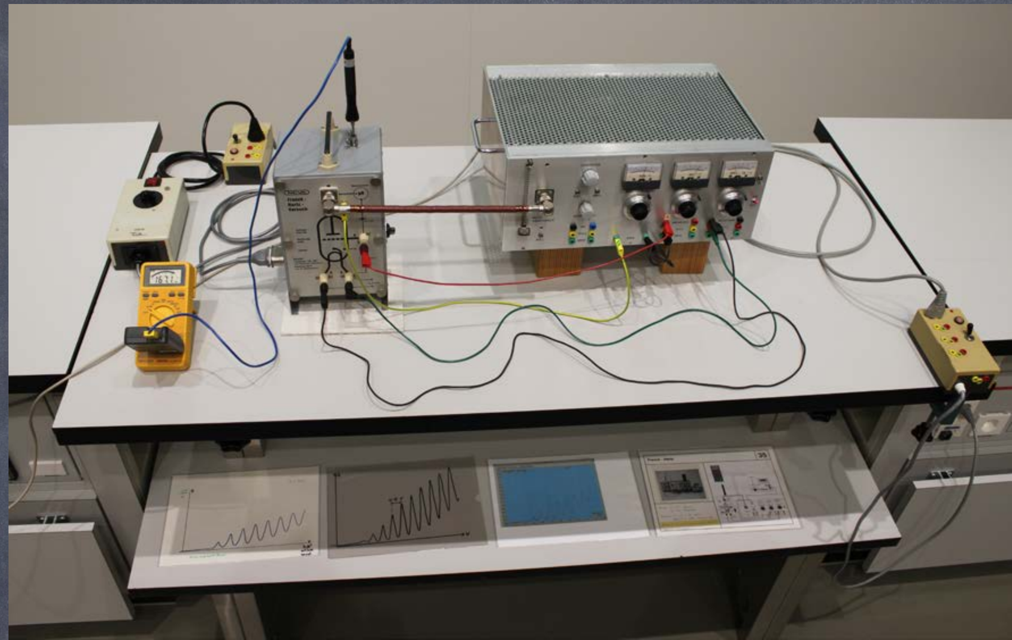


Bohr:
(1913)

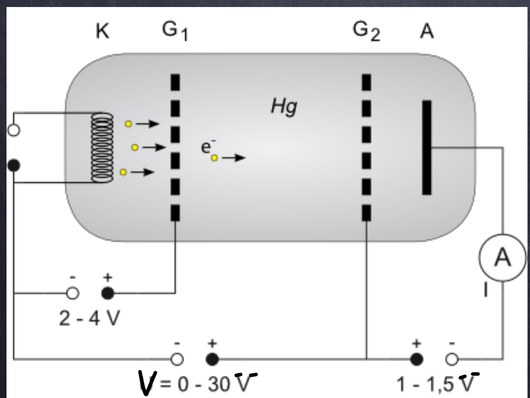


Franck-Hertz experiment (1914) validated the quantum nature theory of atoms





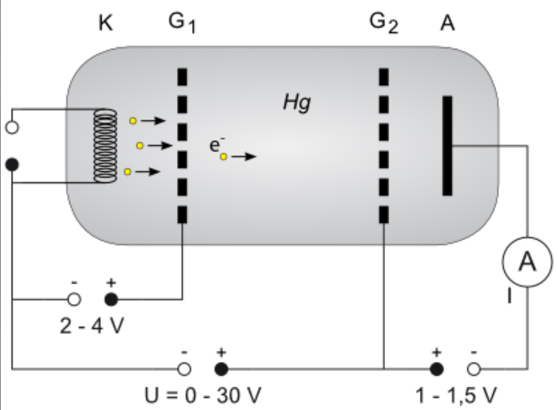
- tube filled with heated mercury vapor.
- Electrons accelerated through tube: $eV = \frac{1}{2}mv^2$ where V changes
- If electrons have at least $(1.5V)e = K$ when passing G_2 they reach the collector and current will be measured in (A)

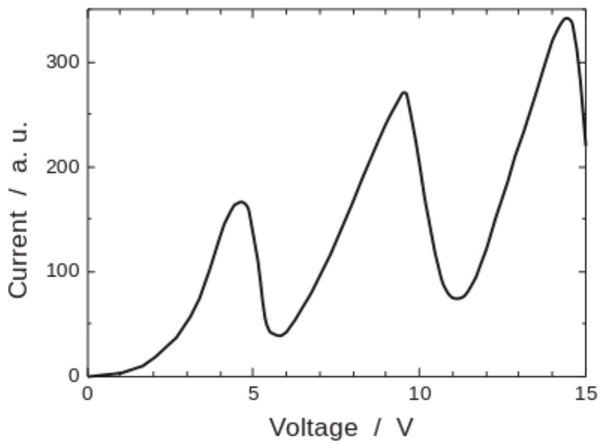


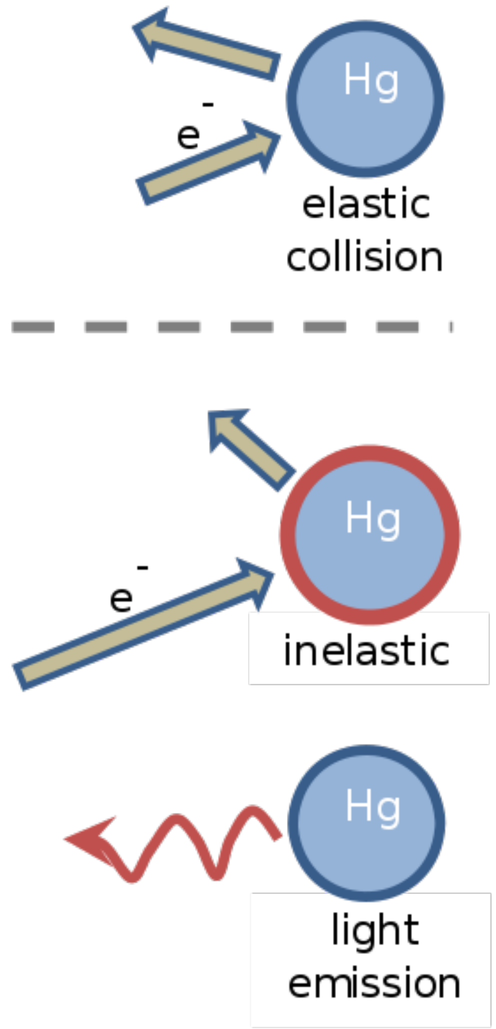
$\therefore \quad I = eAnv$

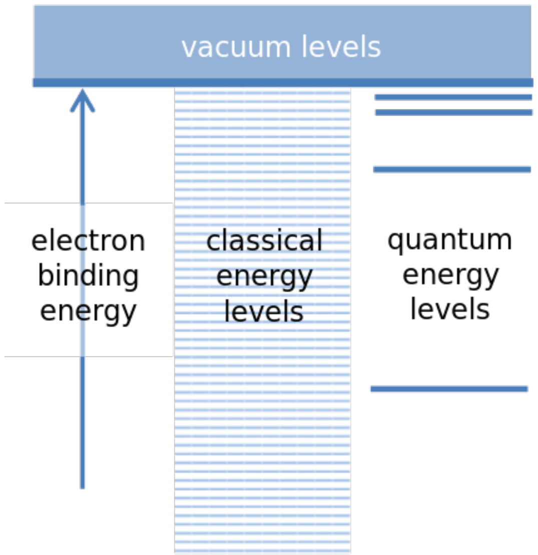
n: charge/volume
A: cross-section area
e: electron charge

Franck-Hertz experiment: shows quantum aspects of atom

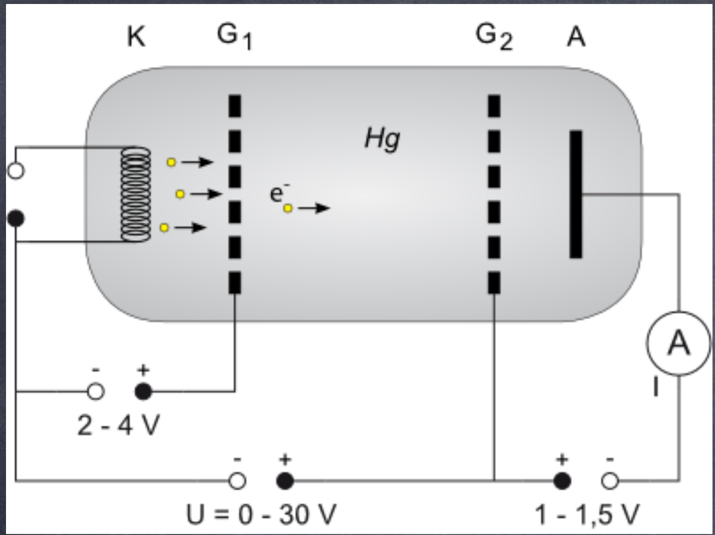


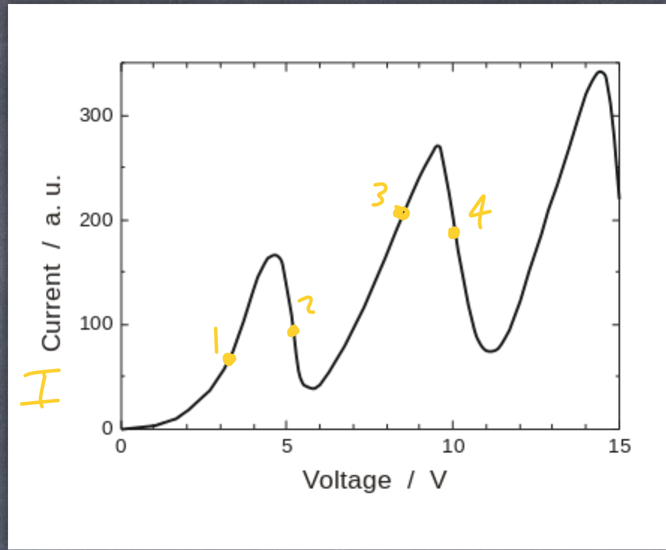


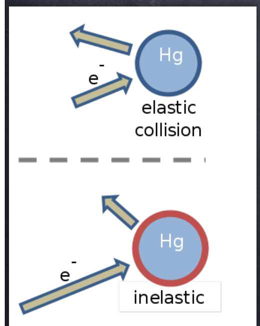




Franck-Hertz eperiment: shows quantum aspects of atom







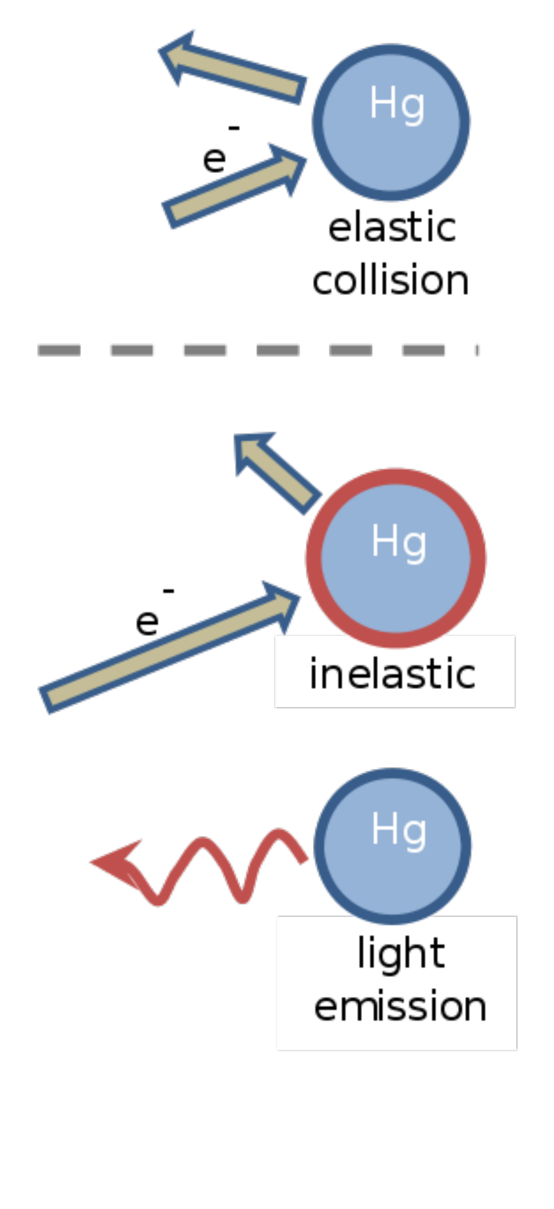
1: V increases → v increases → Current increases

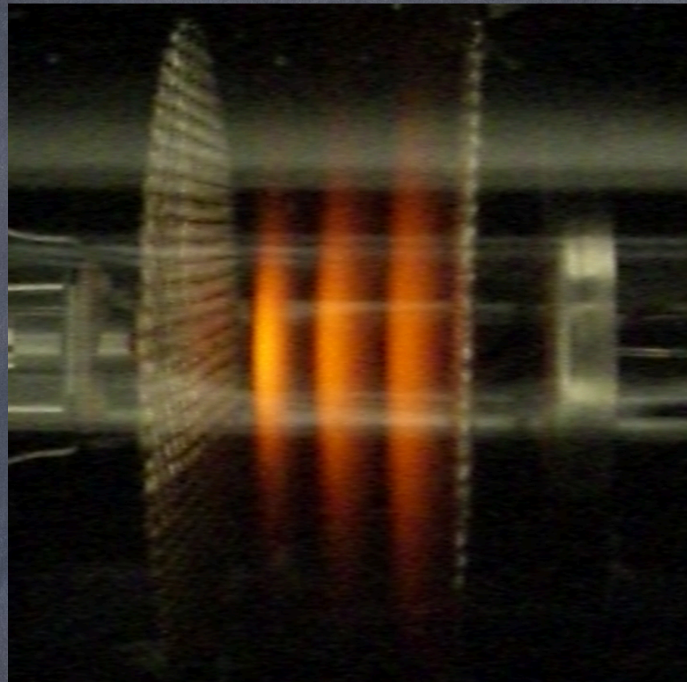
2: Above 4.9 V, atoms can absorb 4.9 eV of electron energy, decreasing v → decreases I

3: Electron has interacted once, but gains v → I increases.

4: Above $2 \cdot 4.9\,V = 9.8\,V$, electron scatters inelastically a second time and loses velocity, → I decreases

If instead of mercury, we would use neon, the





In Coulomb field, $U = \frac{-kZe^2}{r}$ k: Boltzmann constant

For an electron bound to an atom

$$\epsilon = \underbrace{K}_{\text{kinetic energy}} + U = \frac{1}{2}mv^2 - \underbrace{\frac{kZe^2}{r}}_{\text{energy}}$$

In a circular orbit: $F = ma = \frac{mv^2}{r}$ $\frac{v^2}{r}$: centripetal acceleration

$$\underbrace{\frac{mv^2}{r}}_{\text{centripetal force}} = \underbrace{\frac{kZe^2}{r^2}}_{\text{coulomb force}} \Longrightarrow \boxed{\underbrace{\frac{1}{2}mv^2}_{K} = \frac{1}{2}\frac{kZe^2}{r}} \quad (2a)$$

So $$\boxed{\epsilon_r = -\frac{1}{2}\frac{kZe^2}{r}} \quad (2)$$

Energy is a function of r.

Different radii → different energies.

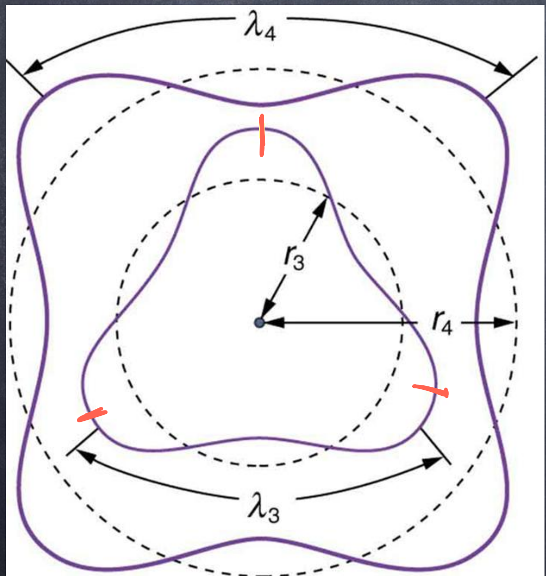
using (1) + (2):

$$\boxed{\nu = \frac{\epsilon_i - \epsilon_f}{h} = \frac{\frac{1}{2}kZe^2}{h}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \quad (3)$$

r_2, r_1: two different radii

Comparing ③ theory with ⓪ expt.,
we see that the radii r_1, r_2 must be proportional to integers squared.

de Broglie considered that an electron orbit around an atom was like standing waves. $\left(\lambda = \frac{h}{p}\right)$



$n\lambda$ = circumference of a circle $= 2\pi r$
for n : integers $= 1, 2, 3, \ldots$

If we take $p = \frac{h}{\lambda}$ (p: momentum)

$\lambda = \frac{h}{p}$ $\left(\hbar = \frac{h}{2\pi}\right)$

$n\lambda = \frac{nh}{p} = 2\pi r \Rightarrow n\hbar = rp = \underbrace{rmv}_{\text{angular momentum.}}$

PHY 117:
$\bar{L} = m\bar{v} \times \bar{r}$
$L = mvr$ for a circle

we see that angular momentum (of electron in atom) is quantized as a result of the standing wave condition:

$$\boxed{n\hbar = mvr} \quad (4)$$

Take (4), square it: $v^2 = \dfrac{n^2\hbar^2}{m^2r^2}$, substitute it into (2a)

$$\frac{1}{2}m\left(\frac{n^2\hbar^2}{m^2r^2}\right) = \frac{kZe^2}{r}$$

solve for r

$$\boxed{r = \frac{n^2\hbar^2}{mkZe^2}} \quad (5)$$

we see that r is quantized.

we define a constant

$$a_0 = \frac{\hbar^2}{mke^2} \approx 0.0529 \text{ nm}$$

which is called the Bohr radius.

Substitute (5) → (3):

$$\nu = \frac{\frac{1}{2}kZe^2}{h}\left(\frac{1}{\frac{n_2^2\hbar^2}{mkZe^2}} - \frac{1}{\frac{n_1^2\hbar^2}{mkZe^2}}\right) \Rightarrow \boxed{\nu = \frac{Z^2mke^4}{4\pi\hbar^3}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)} \quad (6)$$

Compare our theory (6) to our empirical formula (0), the formulas agree, and this constant R is related to other constants, $R = \frac{mk^2e^4}{4\pi c\hbar^3}$

Substitute (5) → (2) :

radius → energy

$$E_n = -\frac{1}{2}\frac{kZe^2}{r} = \frac{-k^2e^4mZ^2}{2\hbar^2}\frac{1}{n^2}$$

n: integer

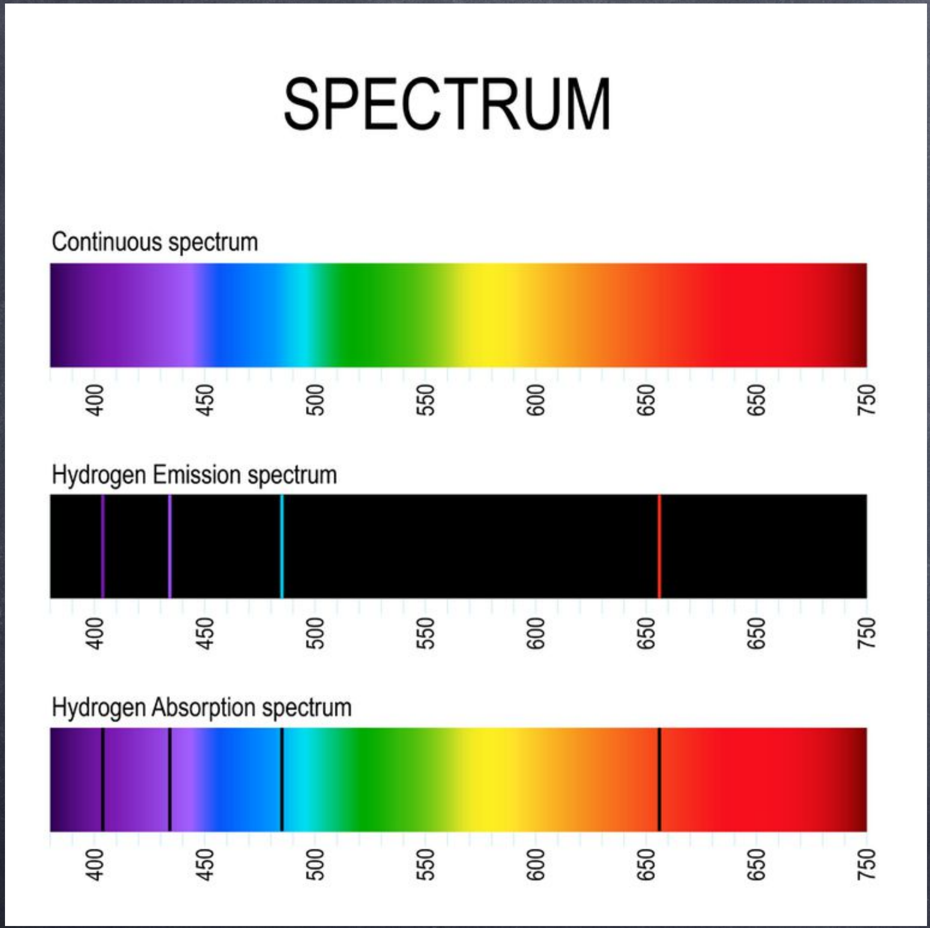
we define ground state energy E_0

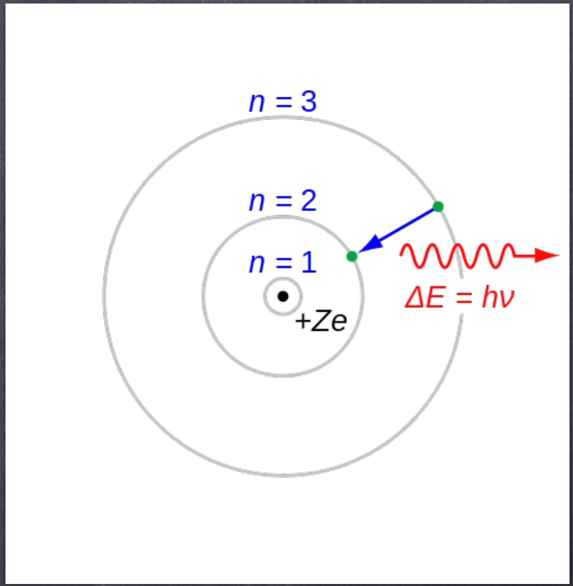
$$\boxed{E_0 = \frac{k^2e^4m}{2\hbar^2} \approx 13.6\,\text{eV}} \quad (7)$$

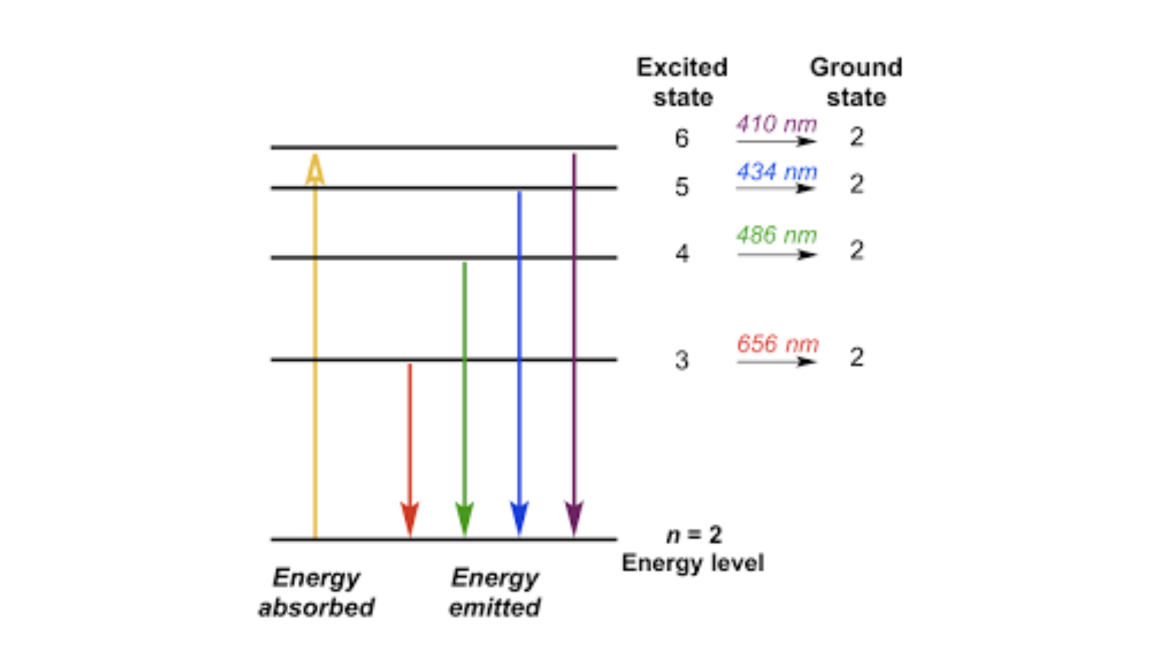
Then

$$\boxed{E_n = \frac{-Z^2}{n^2}E_0} \quad (8)$$

These are the allowed energy levels of the hydrogen atom ($Z=1$)

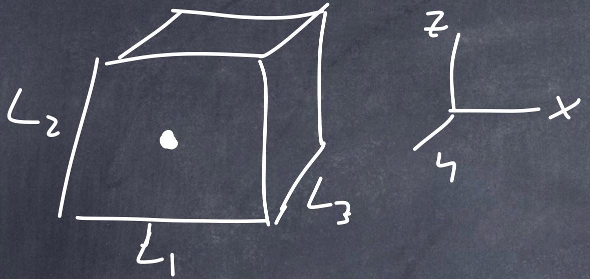

SPECTRUM
Continuous spectrum
400 450 500 550 600 650 650 750
Hydrogen Emission spectrum
400 450 500 550 600 650 650 750
Hydrogen Absorption spectrum
400 450 500 550 600 650 650 750


n = 3
n = 2
n = 1
+Ze
ΔE = hν



Balmer series

A particle trapped in a 3-D box. (extension of our 1-D box)



we use the 3-D Schroedinger equation.

$$\boxed{-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U\psi=E\psi} \quad (1)$$

Here, $U=0$ inside the box, outside $U=\infty$

the wave functions that solve the (1) Factorize:

$$\psi(x,y,z)=\psi(x)\,\psi(y)\,\psi(z)$$

The solution is

$$\boxed{\psi(x,y,z)=A\,(\sin k_1 x)(\sin k_2 y)(\sin k_3 z)} \quad (2)$$

k_1, k_2, k_3 are related to wavelength of our standing waves in each dimension.

A is the constant that is determined by normalizing $1=\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\psi^2(x,y,z)\,dx\,dy\,dz \Rightarrow A$

Insert ② into ①, we find that

$$\epsilon = \frac{\hbar^2}{2m}\left(k_1^2 + k_2^2 + k_3^2\right) \quad + \quad \text{since} \quad p_x = \hbar k_1$$

$$p_y = \hbar k_2$$

$$p_z = \hbar k_3$$

$$\epsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

Look at ②, we need $\psi(x,y,z) = 0$ on the boundaries of our box, when $x = L_1$, $y = L_2$, or $z = L_3$

$x = L_1$: $\sin(k_1 L_1) \overset{\text{must}}{=} 0 \Rightarrow$ true if $k_1 L_1 = n_1 \pi$

so $k_1 = \frac{n_1 \pi}{L_1}$ $\quad n_1$: integer

Likewise: for $y = L_2$: $k_2 = \frac{n_2 \pi}{L_2}$

$z = L_3$: $k_3 = \frac{n_3 \pi}{L_3}$

These are allowed energies of a ptcl in a 3-D box

Finally, we see that

$$\epsilon_{n_1, n_2, n_3} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right)$$

integers n_1, n_2, n_3

If $L_1 = L_2 = L_3$, energies are "degenerate"

$\epsilon_{2,2,1} = \epsilon_{2,1,2} = \epsilon_{1,2,2}$ — $9\epsilon_1$

$\epsilon_{1,1,2} = \epsilon_{1,2,1} = \epsilon_{2,1,1}$ — $6\epsilon_1$

$\epsilon_{1,1,1}$ — $3\epsilon_1$

But if $L_1 \neq L_2 \neq L_3$, then these energies would be split:

$\epsilon_{2,1,2}$

$\epsilon_{2,2,1}$

$\epsilon_{1,2,2}$

$\epsilon_{1,1,2}$

$\epsilon_{1,2,1}$

$\epsilon_{2,1,1}$

$\epsilon_{1,1,1}$

Now, we consider a 3D atom, which has a potential:

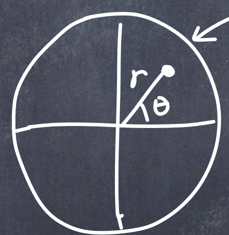
$$U = -\frac{kZe^2}{r}$$

This is a spherical potential of Coulomb field of a atom of charge Z with one electron.

— First consider a 2D sphere.

Because particles behave as waves, to solve where the particle is, & its energies, we have to consider standing wave problem for different boundary conditions.

Drum



border is Fixed.

2-D membrane
(circular)

we have 2 degrees of freedom.

2 "quantum" numbers
$m = 0, 1, 2, \cdots$
$n = 0, 1, 2, \cdots$

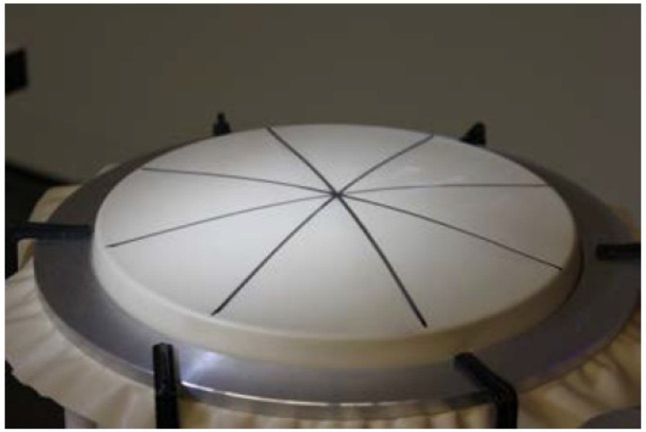
solutions to 2D circle : Bessel functions

$$\Psi(r,\theta) = \Psi(r)\,\Psi(\theta)$$

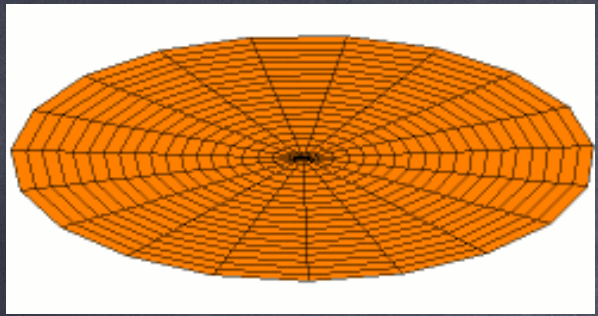
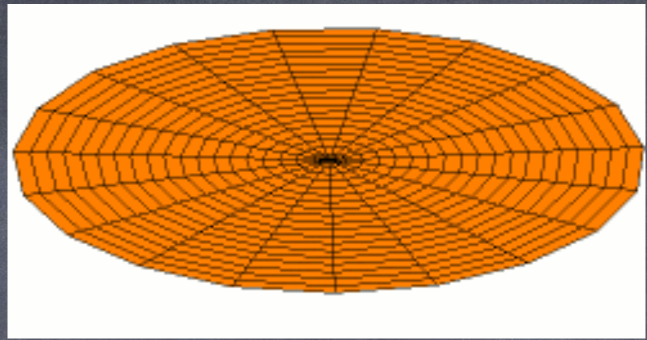
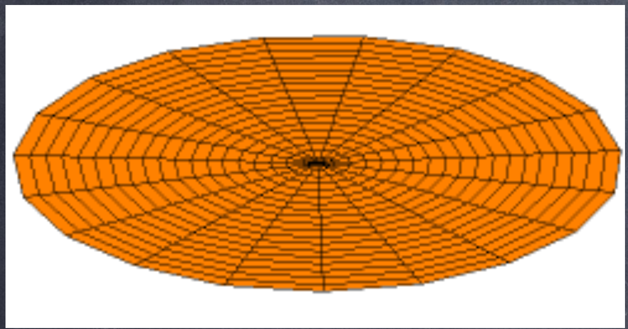
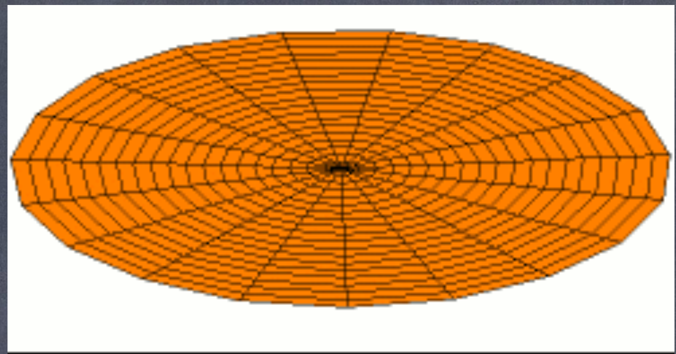
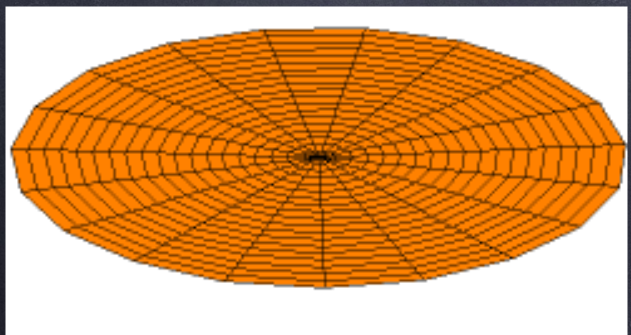
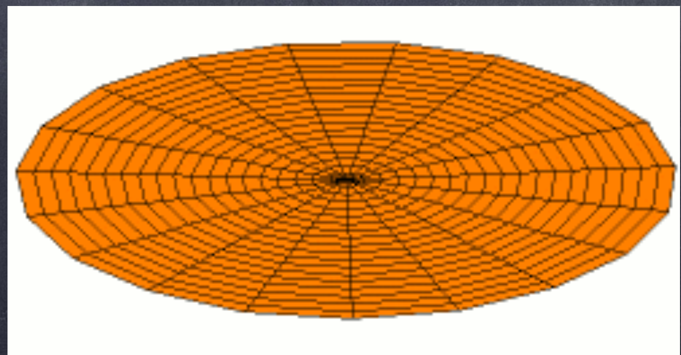
Here Ψ is height of drum membrane

If we add time,

$$\Psi(r,\theta,t) = \Psi(r)\,\Psi(\theta)\,\Psi(t)$$



Grundschwingung bei ca. 31Hz

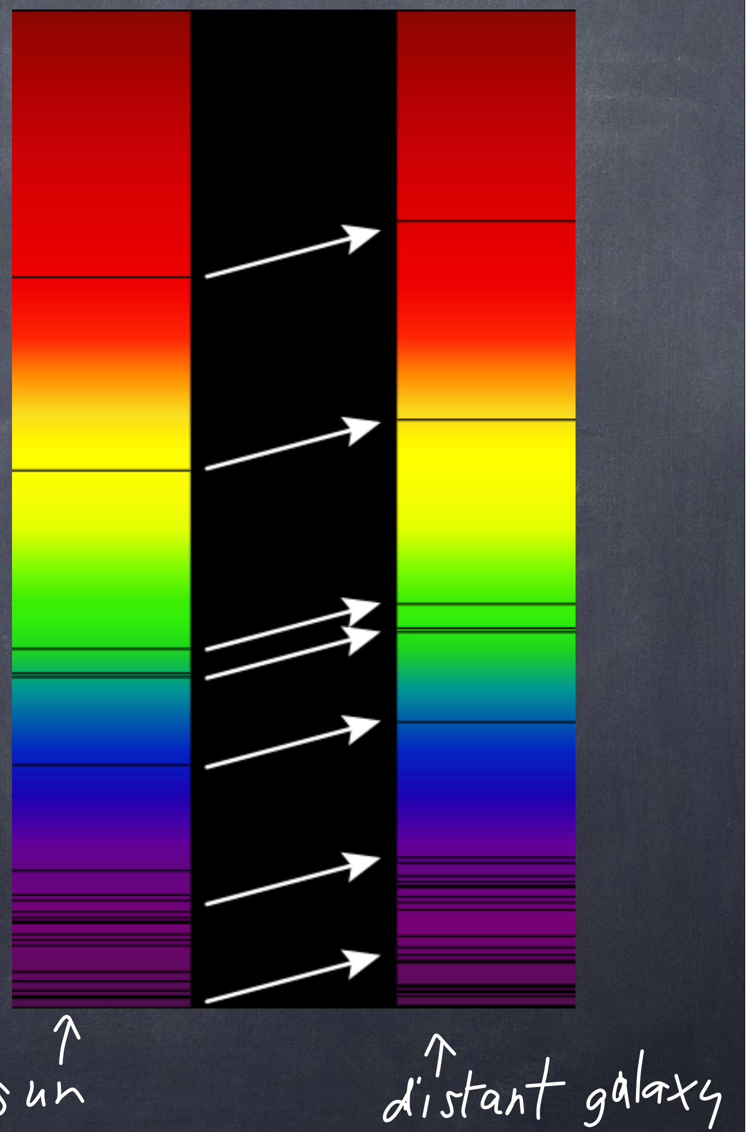
If time, "red shift" of absorption lines

The same spectral lines are shifted due to the relativistic Doppler effect.

This is a shift of λ & ν due to the velocity of an object wrt to us.

Allows scientists to measure speed of distant objects.

Let's us observe that the universe is expanding (accelerating!)



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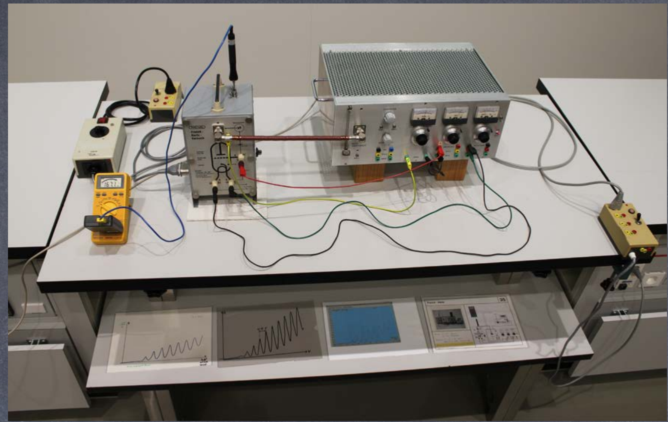

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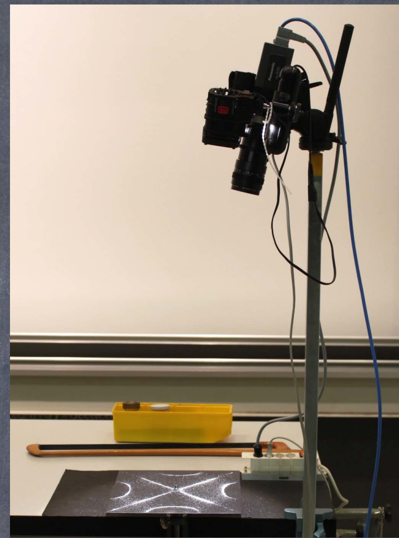

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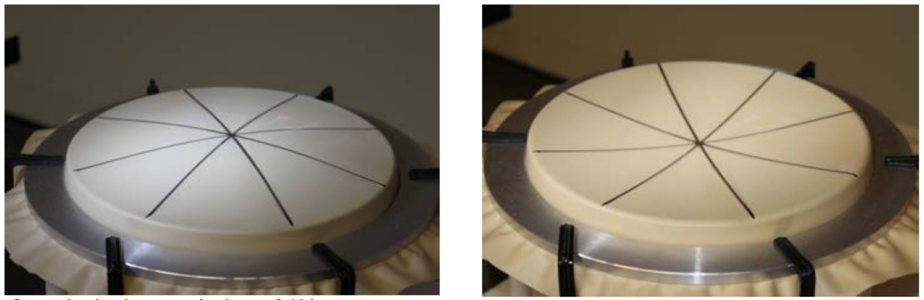
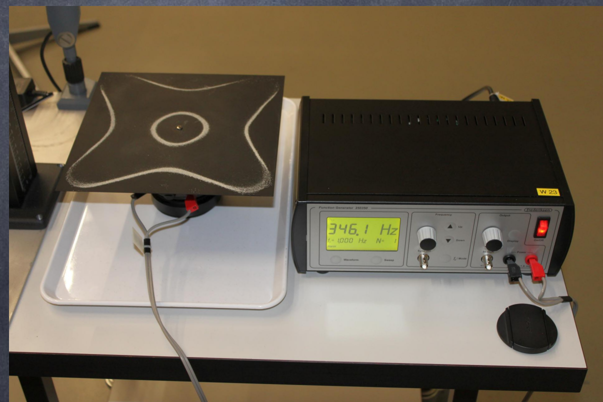


Grundschwingung bei ca. 31Hz

A58



W23