QCD equation of state via the complex Langevin method

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Particle Physics Seminar
ETH Zürich and University of Zürich

Felipe Attanasio[1], Benjamin Jäger[2] and Felix Ziegler[3]

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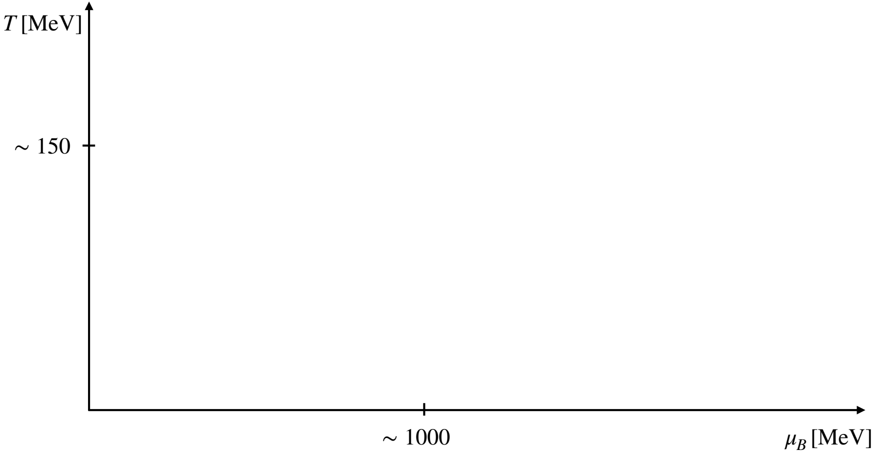
[1] Heidelberg University
[2] University of Southern Denmark (SDU)
[3] The University of Edinburgh

Overview

1. The QCD phase diagram
2. The Sign problem
3. The complex Langevin method
4. Numerical results
5. Conclusion and plans

The QCD phase diagram

Sketching the QCD phase diagram



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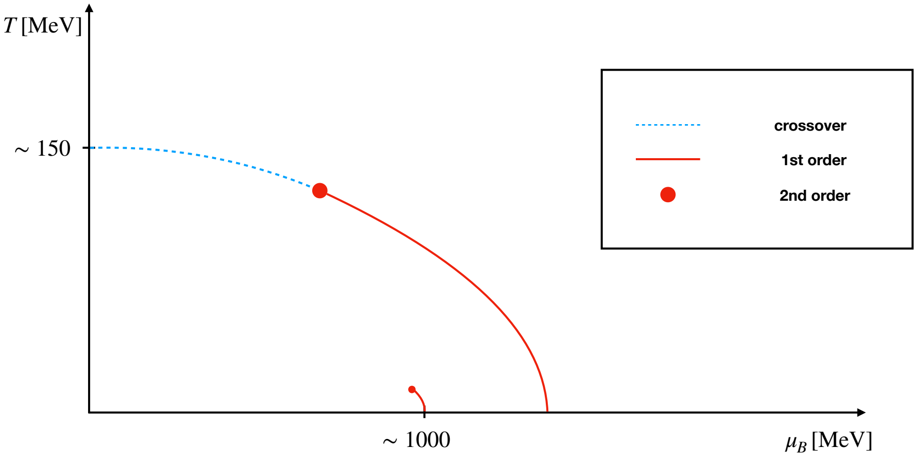

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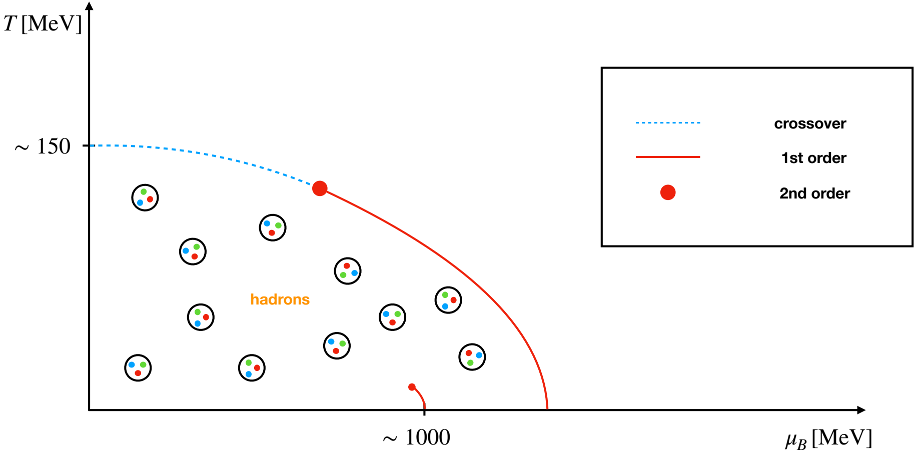

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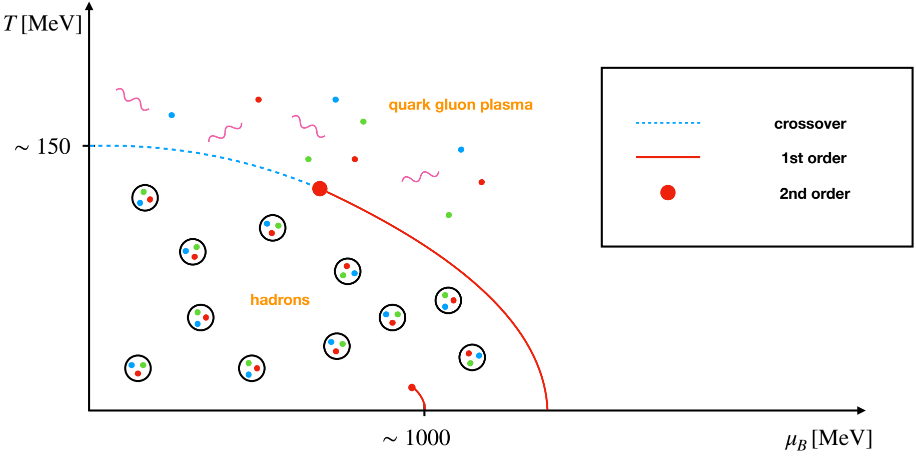

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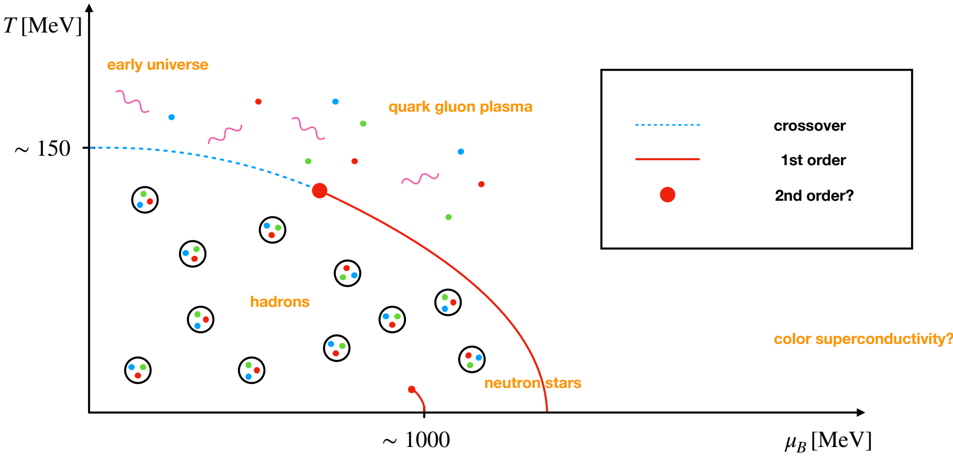
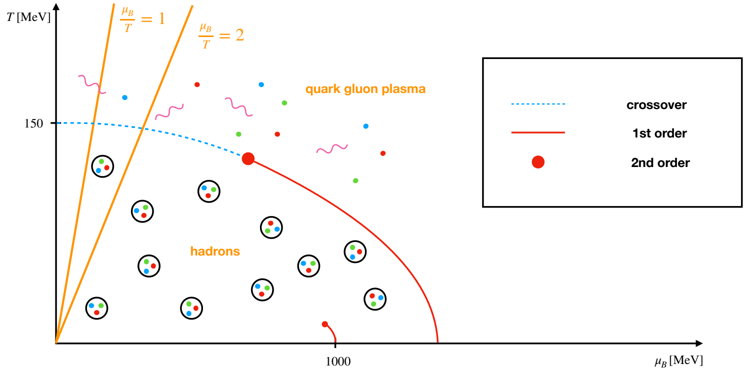
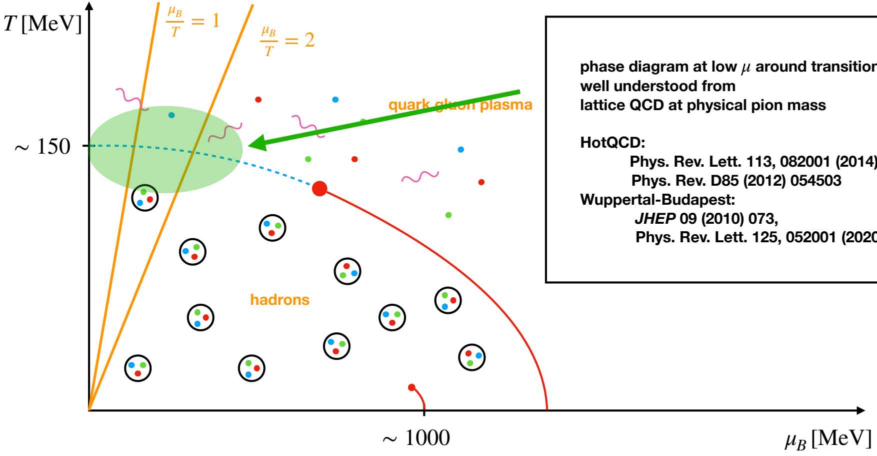


Mapping out the QCD phase diagram

- experiment: heavy-ion collisions LHC, RHIC, FAIR
- theory: **lattice QCD** or functional methods
- finite μ_B restricts conventional lattice Monte Carlo simulations substantially $\rightarrow$ **SIGN PROBLEM**



Status of the QCD phase diagram from lattice QCD



phase diagram at low μ around transition line well understood from lattice QCD at physical pion mass

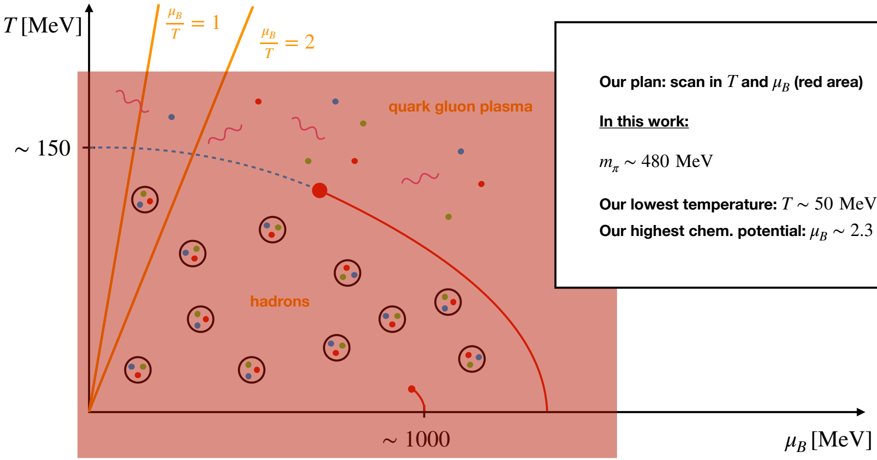
HotQCD:

Phys. Rev. Lett. 113, 082001 (2014),
Phys. Rev. D85 (2012) 054503

Wuppertal-Budapest:

***JHEP* 09 (2010) 073,**
Phys. Rev. Lett. 125, 052001 (2020)

Our plan for lower T and higher μ_B



Our plan: scan in T and μ_B (red area)

<u>In this work:</u>

$m_\pi \sim 480$ MeV

Our lowest temperature: $T \sim 50$ MeV

Our highest chem. potential: $\mu_B \sim 2.3$ GeV

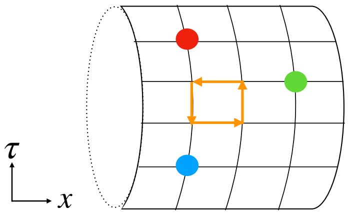
Lattice QCD at finite chemical potential

$$Z[T,\mu] = \int_{SU(3)^\Omega} \mathcal{D}U \exp(-S_{\text{eff}}[U;T,\mu])\,,$$

$$\mu = \mu_B/3\,,\; T = \frac{1}{aN_t}\,,\Omega = N_t N_s^3\,.$$

In this talk:

$\Omega = 32 \times 24^3 = 1769472$



Euclidean action

$$S_{\text{eff}}[U;T,\mu] = -N_f \log(\det M[U;T,\mu]) + \frac{\beta}{3}\sum_{x,\rho<\sigma} \mathrm{Tr}\left[\mathbb{1} - \frac{1}{2}\left(U_{x,\rho\sigma} + U^{-1}_{x,\rho\sigma}\right)\right]$$

A closer look at the fermion matrix

$$M_{xy}[U;T,\mu] = (4+m)\delta_{xy} - \frac{1}{2}\sum_{\nu}\left[\Gamma_{\nu}\, e^{\mu\delta_{\nu,0}}\, U_{x,\nu}\delta_{x+\hat{\nu},y} + \Gamma_{-\nu}\, e^{-\mu\delta_{\nu,0}}\, U^{-1}_{x-\hat{\nu},\nu}\delta_{x-\hat{\nu},y}\right] ,$$

$$\Gamma_{\pm\nu} = 1 \mp \gamma_{\nu}$$

- μ renders $S_{\text{eff}}[U;T,\mu]$ complex
- no conventional Monte Carlo sampling applicable
- How bad is the sign problem?

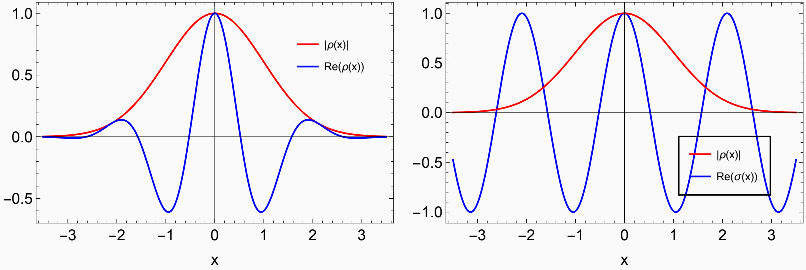
For $U \in SU(3)$

$$M[U;T,\mu]^{\dagger} = \gamma_5 M(-\mu^{*})\gamma_5$$

$$\det M[U;T,-\mu^{*}] = (\det M[U;T,\mu])^{*}$$

The Sign problem

A solvable sign problem?



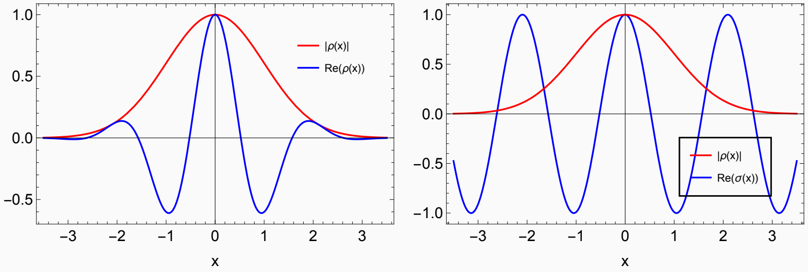
Toy model for a free theory

$$S[x;\lambda] = \frac{x^2}{2} + i\lambda x\,, \quad \lambda \in \mathbb{R}$$

$$Z[\lambda] = \int_{\mathbb{R}} dx \exp(-S[x;\lambda]) = e^{-\lambda^2/2}$$

In the figure: $\lambda = 3$

A solvable sign problem?



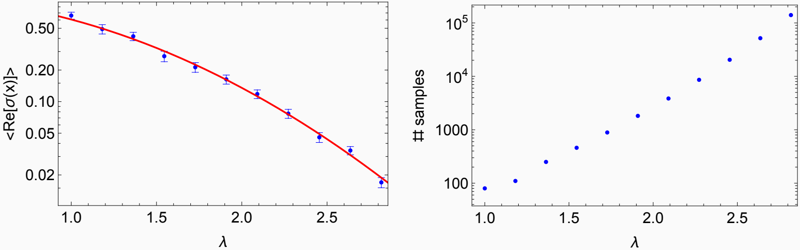
$$\rho(x;\lambda) = \exp(-S[x;\lambda])$$
$$\sigma(x;\lambda) = \exp(-i\mathrm{Im}(S[x;\lambda]))$$

Try to save MC sampling via phase reweighting

$$\langle O(x)\rangle_R = \frac{1}{Z_R}\int_{\mathbb{R}} dx\ O(x)\ |\rho(x;\lambda)| \quad Z_R = \int_{\mathbb{R}} dx\ |\rho(x;\lambda)|$$

In the figure: $\lambda = 3$

A solvable sign problem?

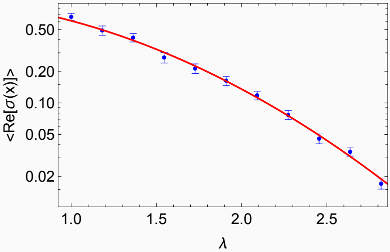


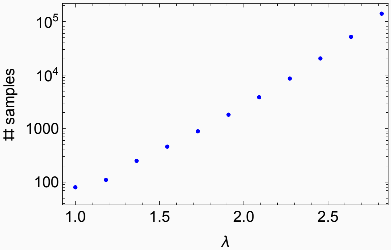
- Perform **phase reweighting**, i.e. a Monte Carlo estimate of

$$Z[\lambda] = \sqrt{2\pi}\langle\sigma(x;\lambda)\rangle_R$$

- samples are drawn from $|\rho(x)| \Rightarrow$ **exponentially hard problem**.

A solvable sign problem?





In QCD cost of phase reweighting increases exponentially with the space-time volume Ω.

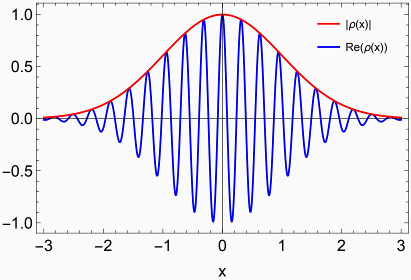
Average phase

$$\langle \sigma \rangle_{\mathrm{PQ}} \propto \exp(-\Omega(F - F_{\mathrm{PQ}})) ,$$

F (F_{PQ}) is free energy of the full (phase-quenched) theory.

Ideas to face the sign problem

In QCD cost of phase reweighting increases exponentially with the space-time volume.



reweighting, Taylor expansions, imaginary μ, dual formulations, density of states, **complex Langevin**, Lefschetz thimbles, flowed manifolds,... De Forcrand, PoS LAT2009, Alexandru et. al., Rev.Mod.Phys. 94 (2022), Attanasio, Jäger, FPGZ, Eur. Phys. J. A 56 (2020), Guenther PoS LAT2021

The complex Langevin method

The complex Langevin method in a nutshell

Stochastic Quantization

Parisi and Wu in Sci. Sin. 24 483 (1981)

- Evolve fields in fictitious time θ (Langevin or computer time)
- Langevin equation

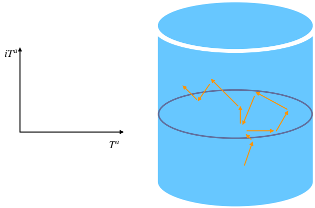
$$\frac{\partial\phi(x,\theta)}{\partial\theta} = -\frac{\delta S}{\delta\phi(x,\theta)} + \eta(x,\theta)$$

- S is the Euclidean action (≥ 0) and η a Gaussian white noise field
- stationary solution of the associated Fokker-Planck equation is the equilibrium distribution e^{-S}
- observable expectation values

$$\langle\mathcal{O}\rangle = \lim_{\theta_{\max}\to\infty} \frac{1}{\theta_{\max} - \theta_{\mathrm{therm}}} \int_{\theta_{\mathrm{therm}}}^{\theta_{\max}} \mathcal{O}(\theta) d\theta \, .$$

The complex Langevin method in a nutshell

- **Analytic continuation** in the field variables, Parisi, Phys. Lett. B, 131 (1983)
- extending **stochastic quantization** from $SU(3)$ to $\mathbb{SL}(3, \mathbb{C})$.



Langevin equation

$$\frac{\partial \phi}{\partial \theta} = -\frac{\delta S}{\delta \phi} + \eta$$

- S is a meromorphic action
- η Gaussian white noise
- θ fictitious time

The complex Langevin method in a nutshell

Langevin equation

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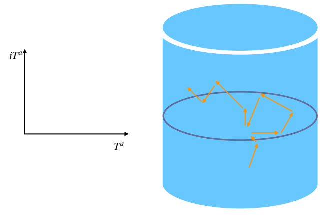


- mathematical foundations and **criteria of correctness**, see
  Seiler et.al., Phys. Lett. B723 (2013),
  Nishimura et.al., Phys. Rev. D 92 (2015),
  Attanasio, Jäger, FPGZ, Eur. Phys. J. A 56 (2020),
  Scherzer et. all, Phys. Rev. D 101 (2020)

Lattice QCD with complex Langevin

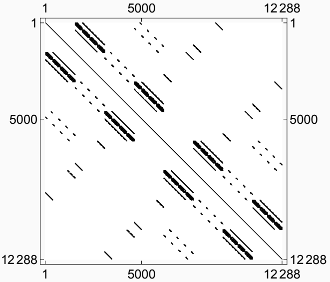
Discretized complex Langevin equation with finite step size ϵ

$$U_{x,\nu}(\theta + \epsilon) = \exp\left[i\epsilon\lambda^a\left(-D^a_{x,\nu}S + \eta^a_{x,\nu}\right)\right] U_{x,\nu}(\theta)\,,$$

$$\langle \eta^a_{x,\nu}\eta^b_{y,\rho}\rangle = 2\delta_{x,y}\delta_{\nu,\rho}\delta_{a,b}\,,\, a = 1,\ldots,8\,.$$

Fermionic drift term

$$-D^a_{x,\nu}S_F = N_f \,\mathrm{Tr}\left[M^{-1}D^a_{x,\nu}M\right]$$



Stabilizing the complex Langevin evolution

Unstable directions in $SL(3, \mathbb{C})$ require to stabilize the CL process in numerical simulations.

Recipe:

(1) Adaptive step size Aarts et. al., Eur. Phys. J. A 49 (2013)

(2) **Gauge cooling:** exploit enlarged gauge freedom: move back to $SU(3)$ by minimizing the unitarity norm $\mathrm{tr}[(UU^\dagger - 1)^2] \geq 0$
Seiler et. al., Phys. Lett. B 723 (2013)

(3) **Dynamic stabilization:** keep CL trajectory close to $SU(3)$ by minimizing imaginary part of the gauge field, caveat:
non-holomorphic modification of the drift term
but disappears as $a \to 0$
Attanasio and Jäger, Eur. Phys. J. C 79 (2019)

Numerical results

Lattice setup

- $\beta = 5.8, \kappa = 0.144, V = 24^3, a \approx 0.06\mathrm{fm}$
  Del Debbio et. al., JHEP 02 (2006)
- $N_f = 2$ Wilson fermions ($c_{\mathrm{SW}} = 0$)
- $m_\pi \approx 480 MeV, m_N \approx 1.3\mathrm{GeV}$
- temperature range: $N_t \in \{64, .., 4\} \leftrightarrow T \in \{50, .., 850\}\mathrm{MeV}$
- We have data for a quark chemical potential range:
  $\mu \in \{0, .., 6500\}\mathrm{MeV}$
- For EOS results focus on $\mu_B \in [0, 1.8 m_N]$ and $T \in [50, 200]\mathrm{MeV}$
- fermion matrix inversion: eoCG

Lowest pion mass and temperatures for $\mu \neq 0$ so far.

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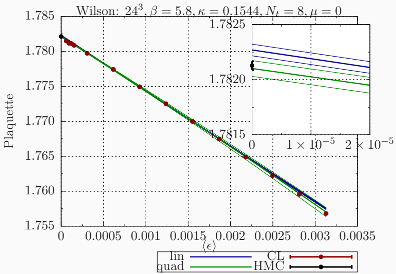
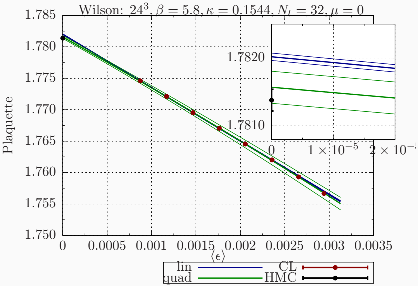
Sanity check

Extrapolation of CL results and comparison with HMC



Deconfined phase



Confined phase

Observables

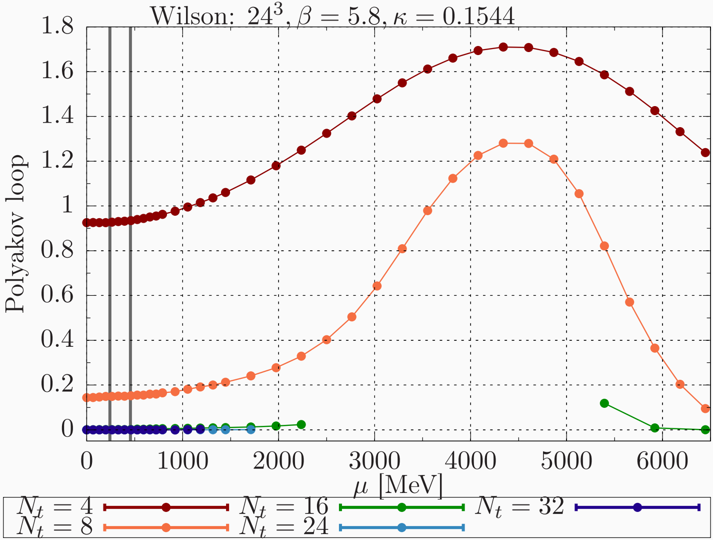
Polyakov loop (confinement - deconfinement transition)

$$P = \frac{1}{3V} \sum_{\vec{x}} \mathrm{Tr} \left\langle \prod_{\tau} U_{(\vec{x},\tau),\hat{0}} \right\rangle$$

Quark density

$$\langle n \rangle = \frac{1}{\Omega} \frac{\partial \log Z}{\partial \mu}$$

Phase diagram on large scales



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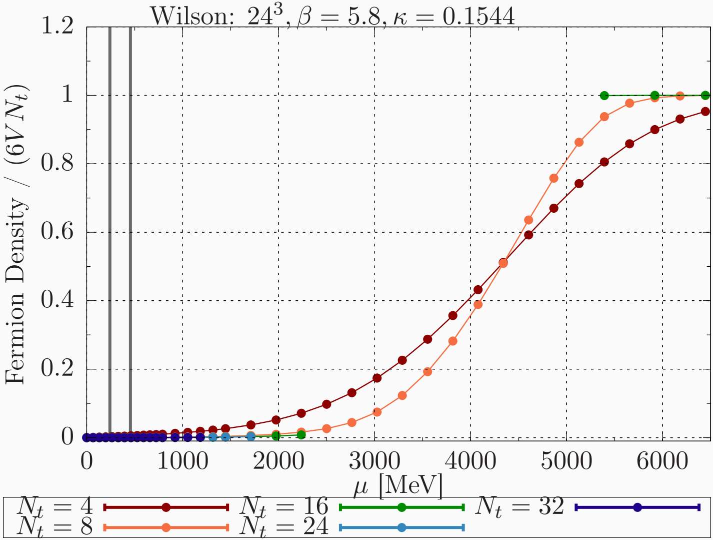
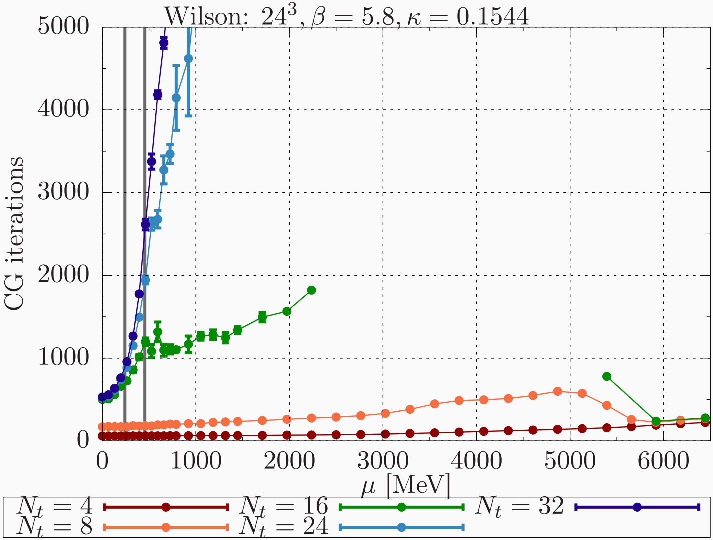
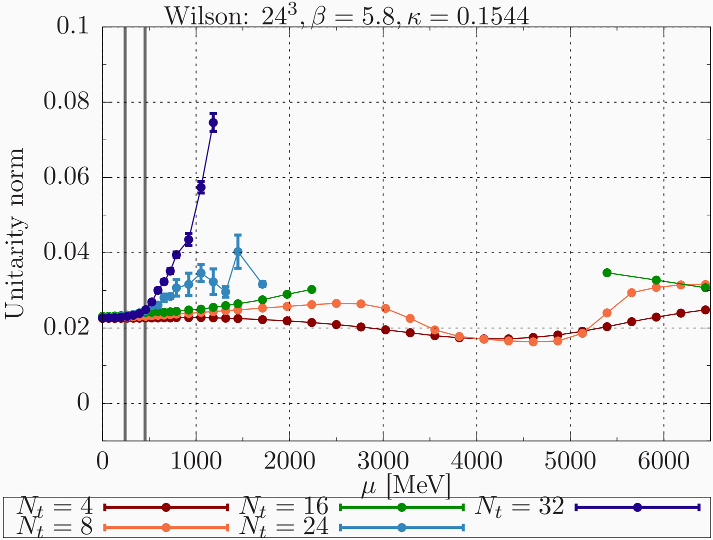


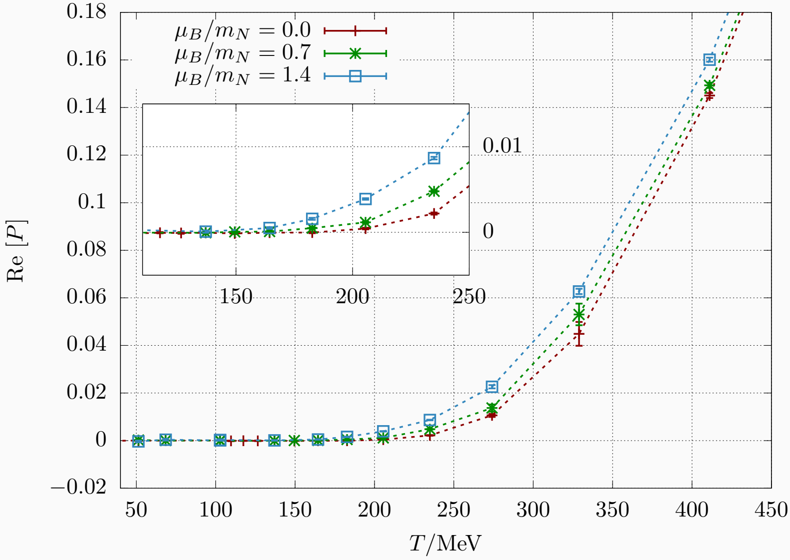
CG iterations



Unitarity norm



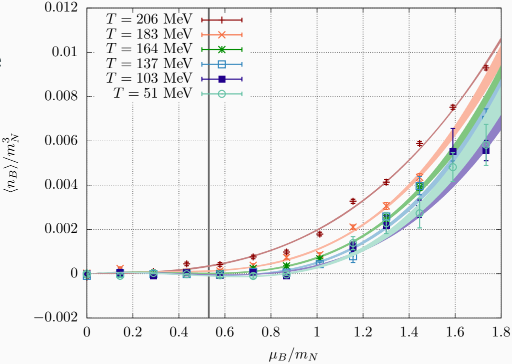
Confinement - deconfinement transition



Phase structure at hadronic scales - baryon density

Silver Blaze phenomenon, T. Cohen, Phys. Rev. Lett. 91 (2003)

- at $T = 0$ expect no μ dependence in thermodynamic observables for $0 \le \mu \le m_N/3$
- grey line indicates $\mu = m_\pi/2$
- Phase-quenched and full theory very different in range $m_\pi/2 < \mu < m_N/3$, severe cancelations of the integrand.



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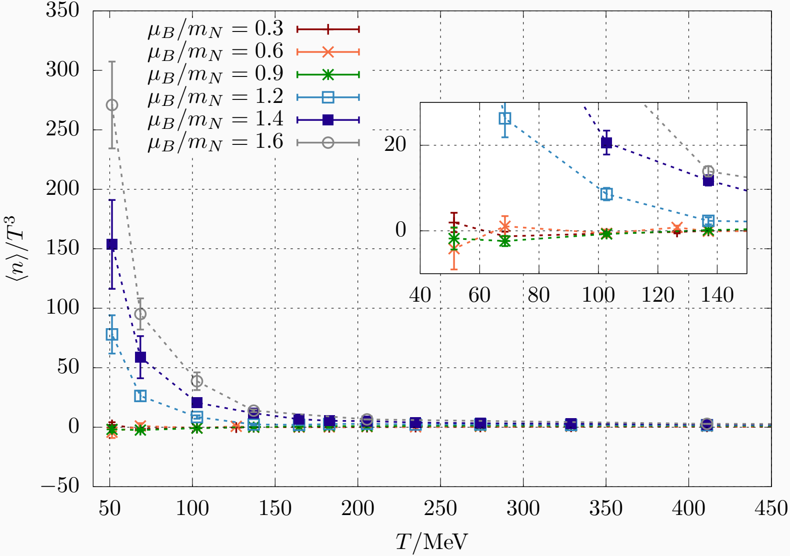
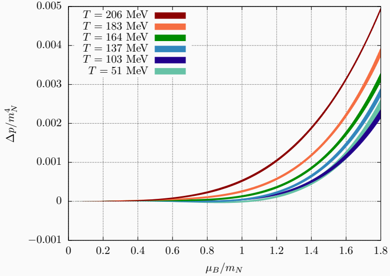
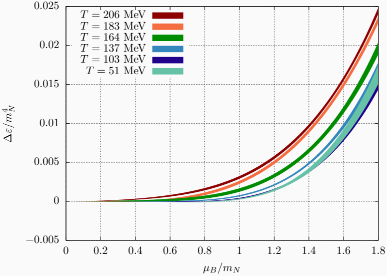


Silver Blaze phenomenon, see T. Cohen, Phys. Rev. Lett. 91 (2003)

Equation of State





Pressure equation of state $\Delta p(\mu_B, T)$ (left) and energy density (right),

$$\Delta p(\mu_B, T) = \int_0^{\mu_B} d\mu' \, \langle n(\mu', T) \rangle$$

Conclusion and plans

Conclusions and ToDo list

Result

- first step towards low temperatures and physical pion mass: **indications of the Silver Blaze phenomenon found.**
- predictions for the QCD equation of state at densities $n \sim 15 n_0$ where n_0 is the nuclear density
- Found that EoS gets stiffer as T decreases.

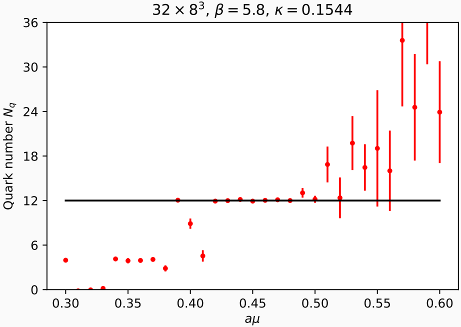
Future plans

- refined mapping of the confinement and chiral transitions, **finite size scaling** $\rightarrow$ critical endpoint (?)
- finer lattices and improved actions (Wilson clover)
- including the strange quark (2+1 flavour simulations)
- **better solvers for the fermionic inversion**
- improvements of systematics related to the CL method (step size extrapolation), criteria of correctness, boundary terms, ...

Thank you very much!

Quarks in a small box

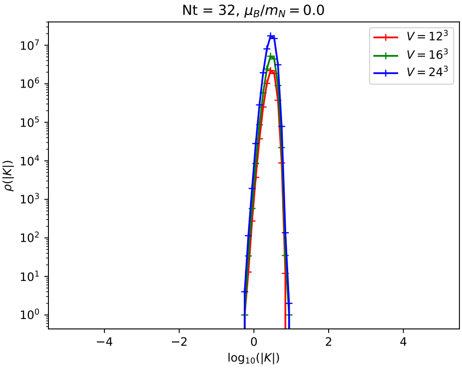
- $V = 8^3,\ T \approx 100\mathrm{MeV}$
- on the lattice expect plateau behaviour of the quark number (H. Matsuoka and M. Stone, Phys. Lett. 136B (1984)
- quantitative agreement between our dynamically stabilized simulations and such based on gauge cooling only (higher pion mass), see PhD thesis by M. Scherzer, Heidelberg, 2019



PRELIMINARY

Examining criteria of correctness

- exponentially decaying histogram of the drift K indicates that CL method works correctly
  Nishimura et.al., Phys. Rev. D 92 (2015)
- We find well localized distributions.



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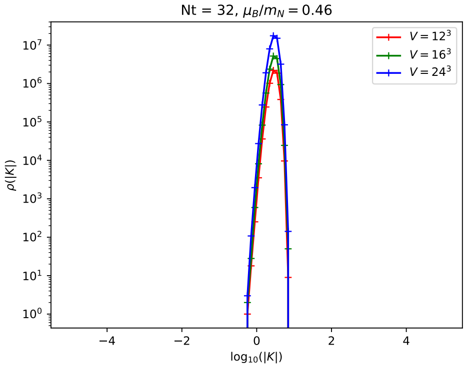

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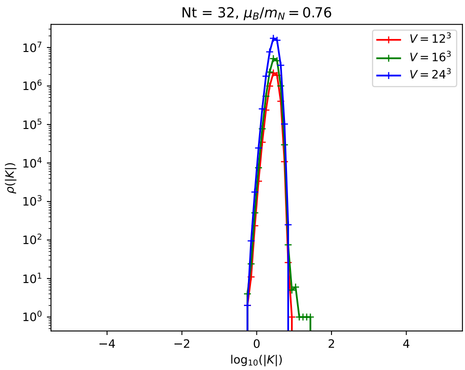

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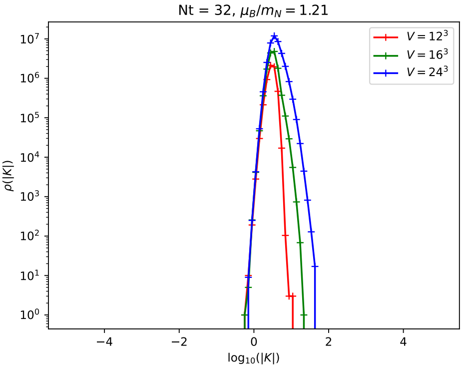

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