BERNHARD MISTLBERGER

FOUR-GLUON SCATTERING AT THREE LOOPS, INFRARED STRUCTURE, AND THE REGGE LIMIT

with Johannes M. Henn and Volodya A. Smirnov

MOTIVATION

- Our capabilities to make precision predictions for the LHC rely on the continuous development of perturbative methods.
- Better understanding of the structure of quantum field theory can lead to improved techniques for calculation.
- Improved techniques for computation can be used to analyze formal aspects of QFT.

A fruitful interplay

MAXIMALLY SUPERSYMMETRIC YANG MILLS THEORY

- Non-abelian gauge theory
- Fields: Gluons, 4 complex fermions, 6 scalars ; all in the adjoint representation of SU(N).
- Vanishing beta-function:
  Free of UV divergences,
  conformal symmetry.

$$\beta = 0$$

- Enhanced degree of symmetry
  (Dual super-conformal symmetry)

MAXIMALLY SUPERSYMMETRIC YANG MILLS THEORY

- Idealised system: Allows for very high order / high multiplicity computations:
    - Hexagon Wilson loop amplitude
      ~ planar 6-point amplitude to 5 loop order
      [Caron-Huot,Dixon,McLeod,Hippel]
    - All order formulae for planar four and five point amplitudes. [Anastasiou, Bern, Dixon, Kosower;Bern, Dixon, Smirnov; Drummond, Henn, Korchemsky, Sokatchev]
- Explicit computation of 2 loop planar amplitudes (and integrands) for 4 and 6 points has lead to deeper understanding of structure of N=4 SYM and QFT in general.

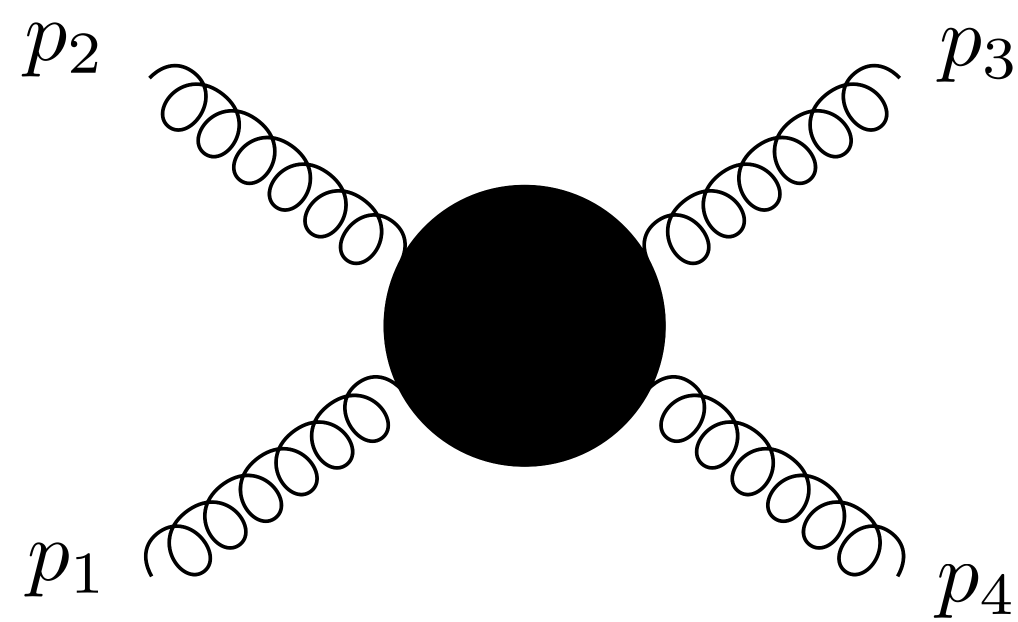
STATUS OF LOOPS AND LEGS

- Data on non-planar amplitudes is scarce (in any theory).
- QCD:
  - 3 loops: Form Factor
  - 2 loops: 4-point.
    Some bits and pieces for 5-points
  - Amplitudes with internal masses are even more difficult.

→ Let's add a data point for 4 legs and 3 loops

4-POINT AMPLITUDE

N=4 SYM AMPLITUDE FOR SCATTERING OF 4 PARTICLES



- Mandelstam invariants:

$$t = (p_2 + p_3)^2$$
$$s = (p_1 + p_2)^2$$
$$u = (p_1 + p_3)^2 = -s - t$$

- On-shell:

$$p_i^2 = 0$$

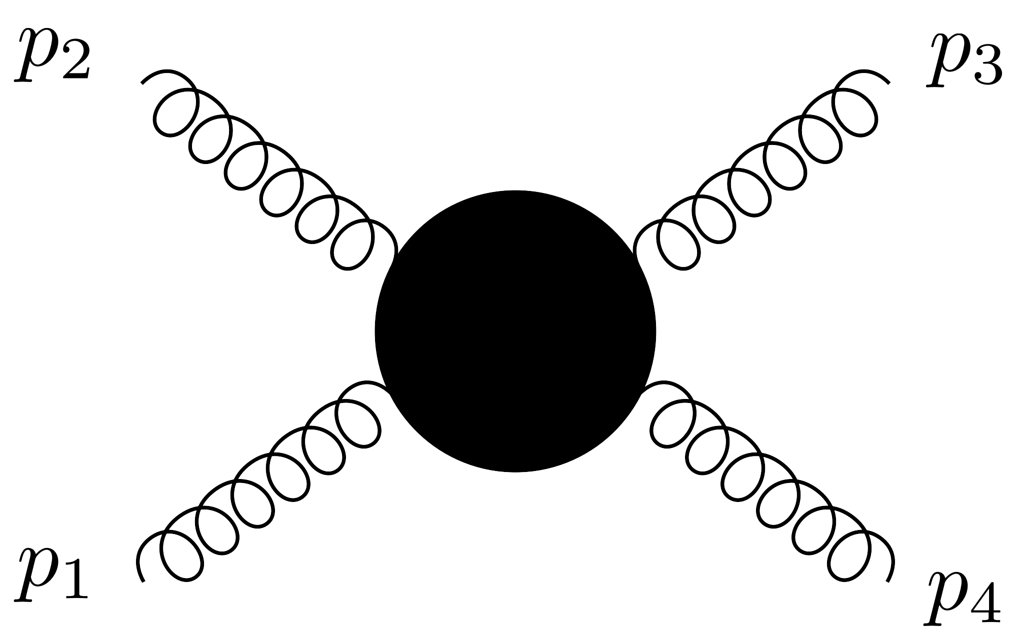
- Perturbative expansion: $\alpha = \frac{g^2}{4\pi^2}(4\pi e^{-\gamma_E})^\epsilon$

$$\mathcal{A}(p_i;\epsilon) = \mathcal{K} \sum_{L=0}^{\infty} \alpha^L \mathcal{A}^{(L)}(s,t;\epsilon).$$

Helicities

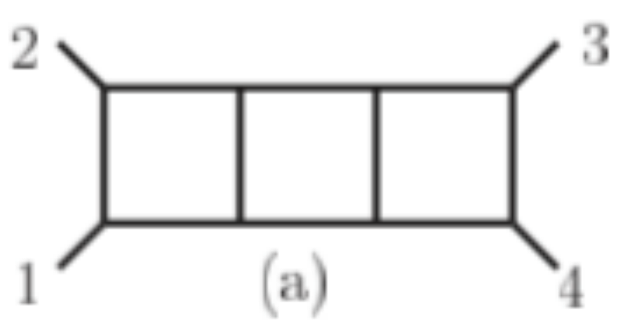
HOW TO CONSTRUCT AN INTEGRAND

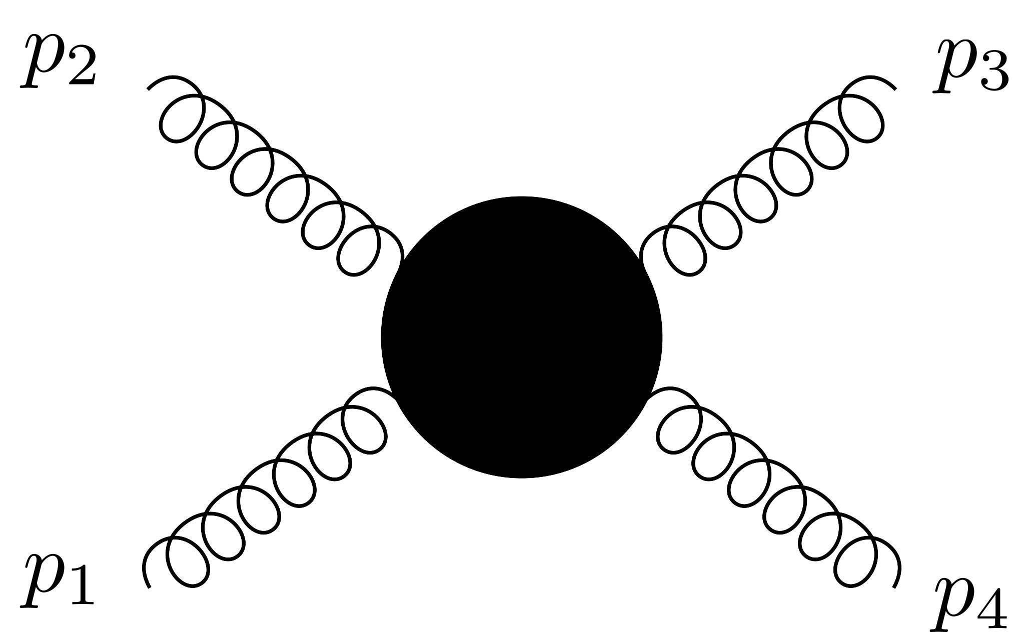
- The number one go-to method: **Feynman Diagrams**



HOW TO CONSTRUCT AN INTEGRAND

- The number one no-go method: **Feynman Diagrams**
- QCD: ~ 80.000 diagrams
- Lots of gauge redundancy
- Naively:
  8 powers of momenta
  in the numerator





HOW TO CONSTRUCT AN INTEGRAND

- Generalised Unitarity Methods

  [Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007]

- Imposing BCJ

  [Bern, Carrasco, Dixon, Johansson, Roiban 2012]

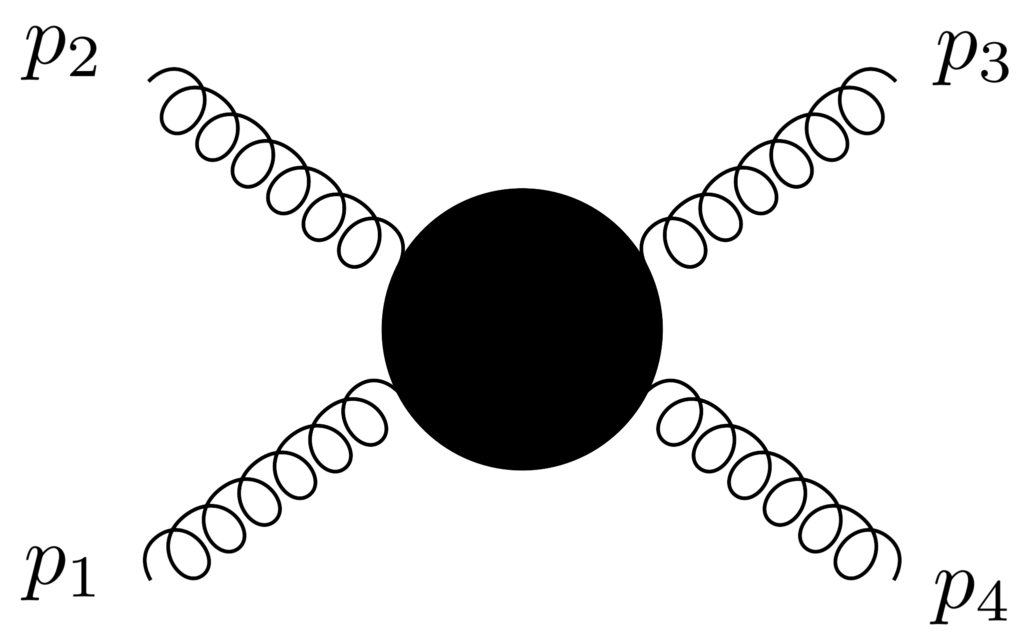
- Manifest UV properties

  [Bern, Carrasco, Dixon, Johansson, Roiban 2008]

- dLog - Forms, No-Poles at Infinity

  [Arkani-Hamed, Bourjaily, Cachazo, Trnka 2014]

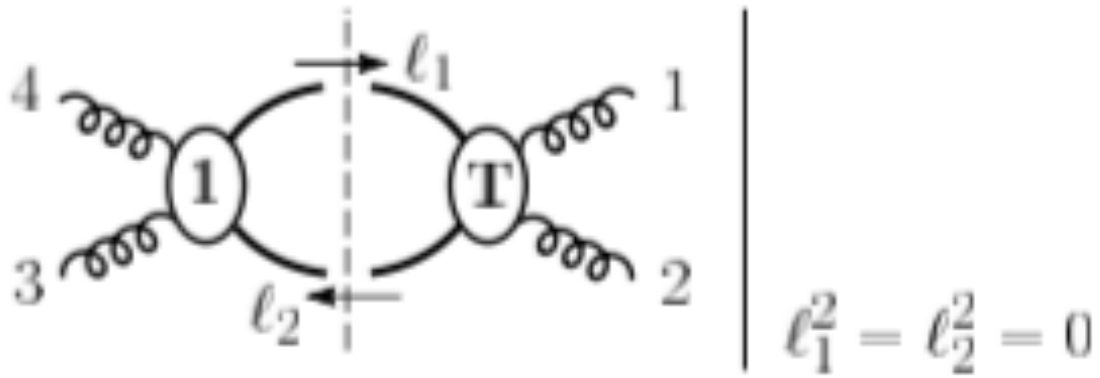
  [Bern, Herrmann, Litsey, Stankowicz, Trnka 2015+2016]



GENERALISED UNITARITY

- Step 1: Make an Ansatz for your amplitude
- Step 2: Constrain and verify your Ansatz by taking iterative cuts of your amplitude

2 Loops:



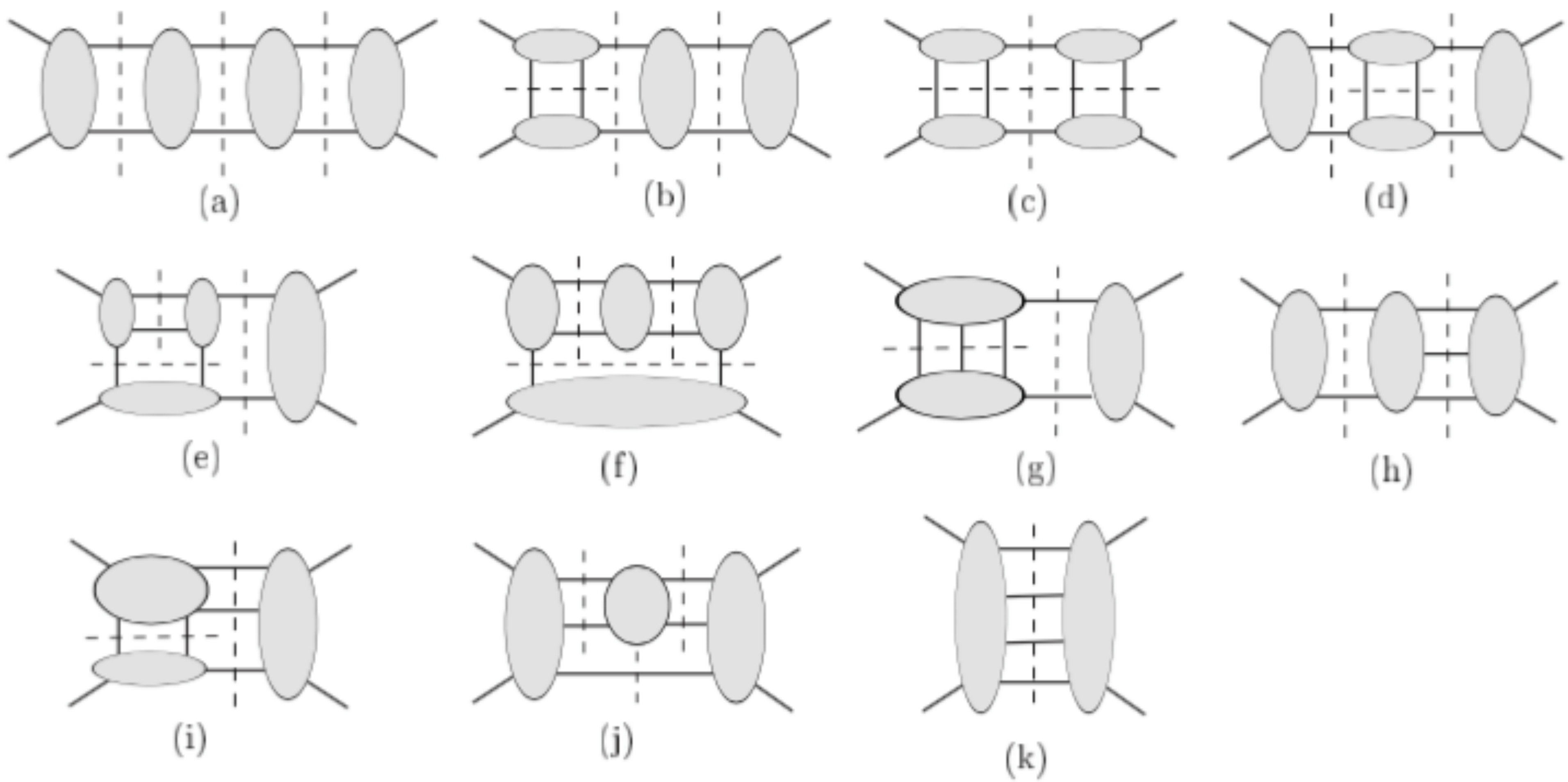
(a)

$$\mathcal{A}_4^{\text{2-loop}}(1,2,3,4)\Big|_{\text{cut(a)}} = \int \sum_{P_1,P_2} \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell_2^2} \mathcal{A}_4^{\text{1-loop}}(-\ell_2,3,4,\ell_1) \frac{i}{\ell_1^2} \mathcal{A}_4^{\text{tree}}(-\ell_1,1,2,\ell_2)\Big|_{\ell_1^2=\ell_2^2=0}$$

[Bern,Rozowsky,Yan]

GENERALISED UNITARITY

- Iterated 2 particle cuts to constrain the amplitude



[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban]

GENERALISED UNITARITY

- Step 1: Make an Ansatz for your amplitude
- Step 2: Constrain and verify your Ansatz by taking iterative cuts of your amplitude
- Caveat: Method has to be valid in $D=4-2\epsilon$ dimensions
  - SuSy-power-counting
  - D-Dimensional Cuts
  - Polarisation sums in D=10, N=1 SYM

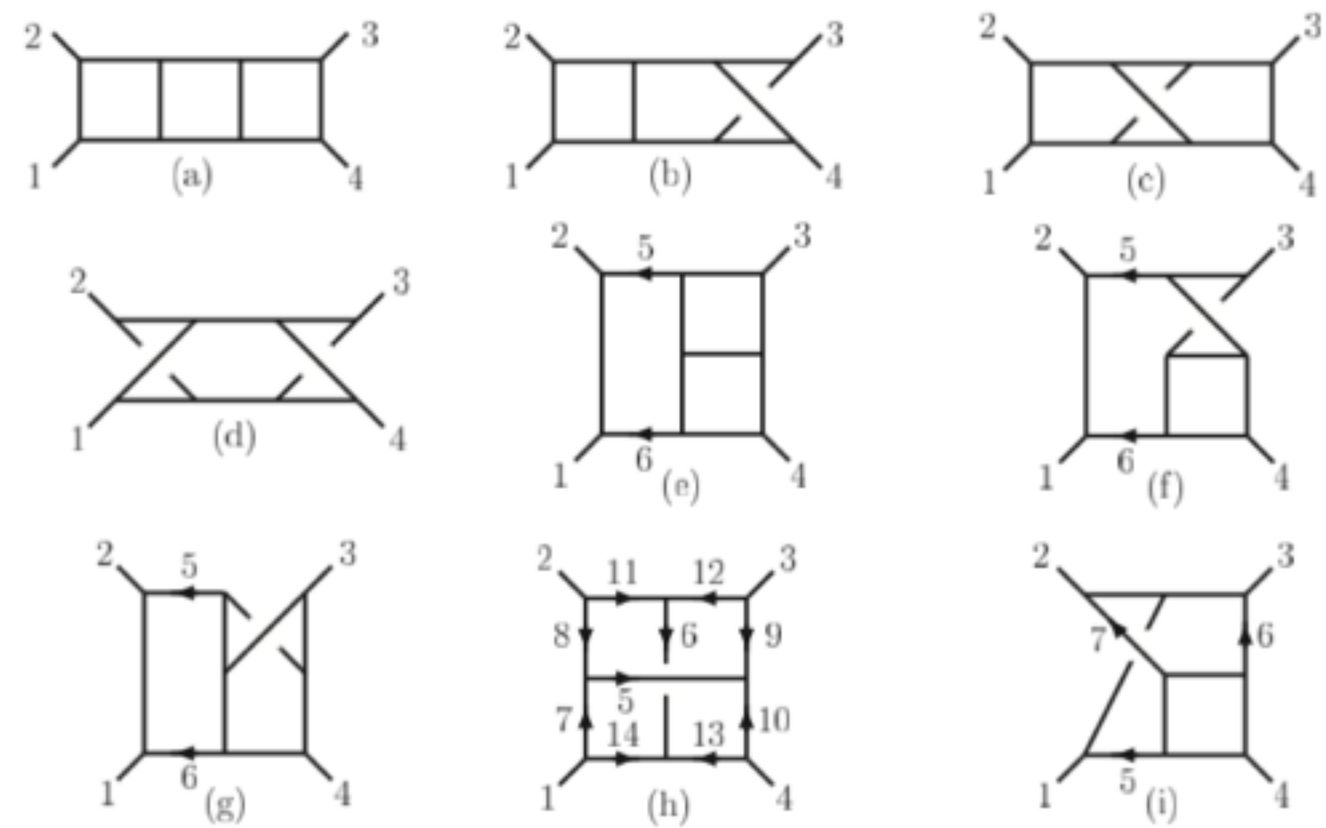
GENERALISED UNITARITY

- Result is an amplitude of remarkable simplicity
  - 3 Loop amplitude can be written in terms of only 9 different Feynman Integrals
  - Symmetric permutation over external legs

$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 s_{12}s_{13}s_{14}M_4^{\text{tree}} \sum_{S_3} \Big[I^{(a)} + I^{(b)} + \tfrac{1}{2}I^{(c)} + \tfrac{1}{4}I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \tfrac{1}{2}I^{(h)} + 2I^{(i)} \Big] .$$

GENERALISED UNITARITY

- 10 - propagator Integrals
- Numerators for Integrals have very low powers of loop-momenta! Huge simplification w.r.t. naive Feynman diagram approach.
- Colour-Factors associated with graphs.



Integral $I^{(x)}$	$N^{(x)}$ for $\mathcal{N}=4$ Super-Yang-Mills
(a)–(d)	s_{12}^2
(e)–(g)	$s_{12}\, s_{46}$
(h)	$s_{12}(\tau_{26}+\tau_{36})+s_{14}(\tau_{15}+\tau_{25})+s_{12}s_{14}$
(i)	$s_{12}s_{45}-s_{14}s_{46}-\frac{1}{3}(s_{12}-s_{14})l_7^2$

AMPLITUHEDRON CONNECTION

[Arkani-Hamed et al]

- Geometric construction for planar N=4 SYM amplitudes: All amplitude integrands take the form

$$dA = \frac{df_1}{f_1} \ldots \frac{df_n}{f_n} \delta(C(f_i).\mathcal{W})$$

- d-Log Form
  - Example: Box Integral ~ 1 Loop Amplitude:

$$d\mathcal{I}_4 = d^4\ell_5 \frac{st}{\ell_5^2(\ell_5 - k_1)^2(\ell_5 - k_1 - k_2)^2(\ell_5 + k_4)^2}$$

$$d\mathcal{I}_4 = \text{dlog}\frac{\ell_5^2}{(\ell_5 - \ell_5^*)^2} \wedge \text{dlog}\frac{(\ell_5 - k_1)^2}{(\ell_5 - \ell_5^*)^2} \wedge \text{dlog}\frac{(\ell_5 - k_1 - k_2)^2}{(\ell_5 - \ell_5^*)^2} \wedge \text{dlog}\frac{(\ell_5 + k_4)^2}{(\ell_5 - \ell_5^*)^2}$$

[Bern, Herrmann, Litsey, Stankowicz, Trnka]

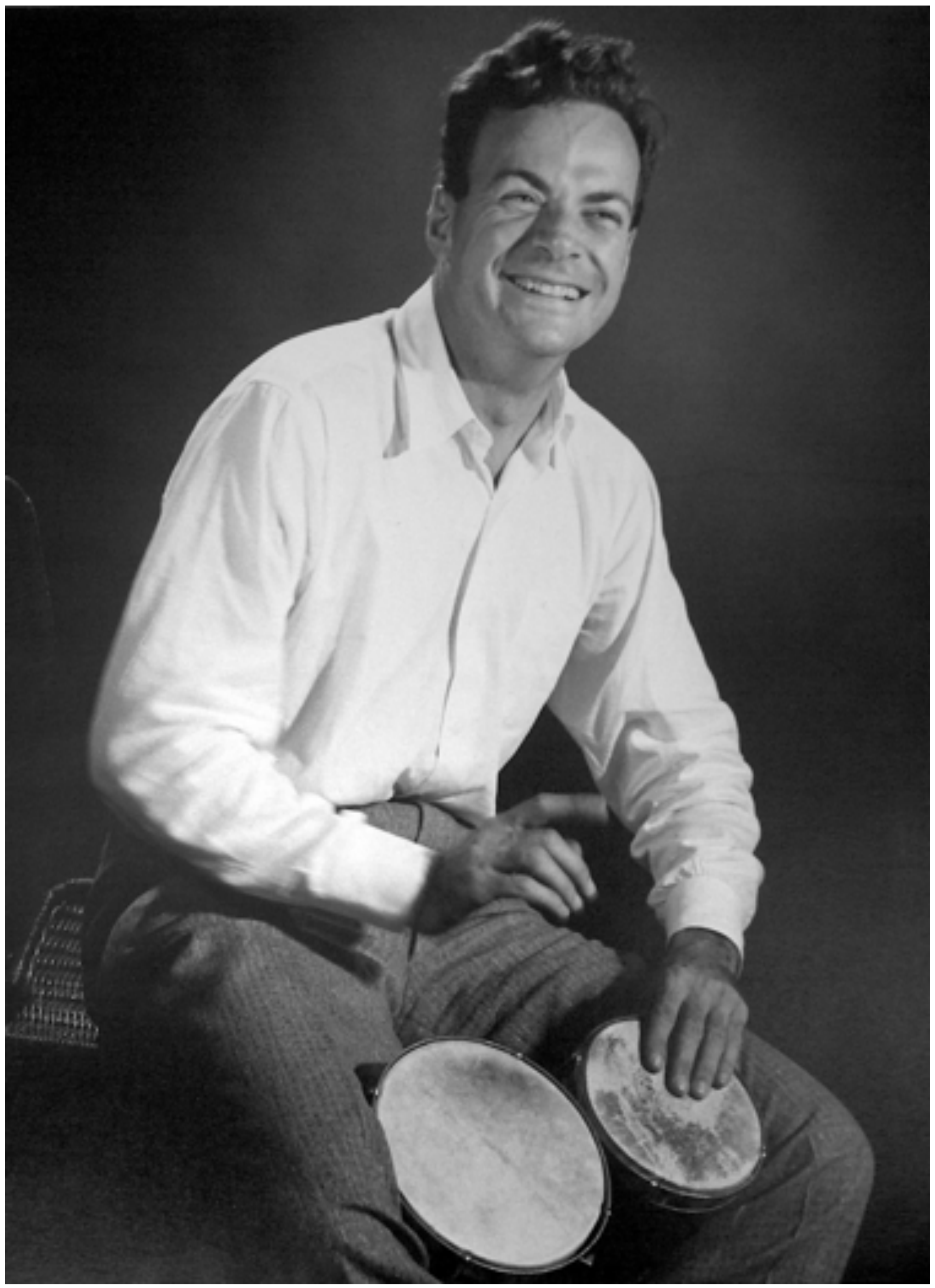
$l_5 \to \infty$: no pole!

AMPLITUHEDRON CONNECTION

- Amplituhedron only known for planar N=4 SYM
- Never mind: Let's say the similar properties hold for non-planar N=4 SYM
  - Only d-Log Integrals
  - No pole as $l_i \to \infty$
- Find basis integrals that satisfy this properties and express the integrand in this basis.

[Bern, Herrmann, Litsey, Stankowicz, Trnka]

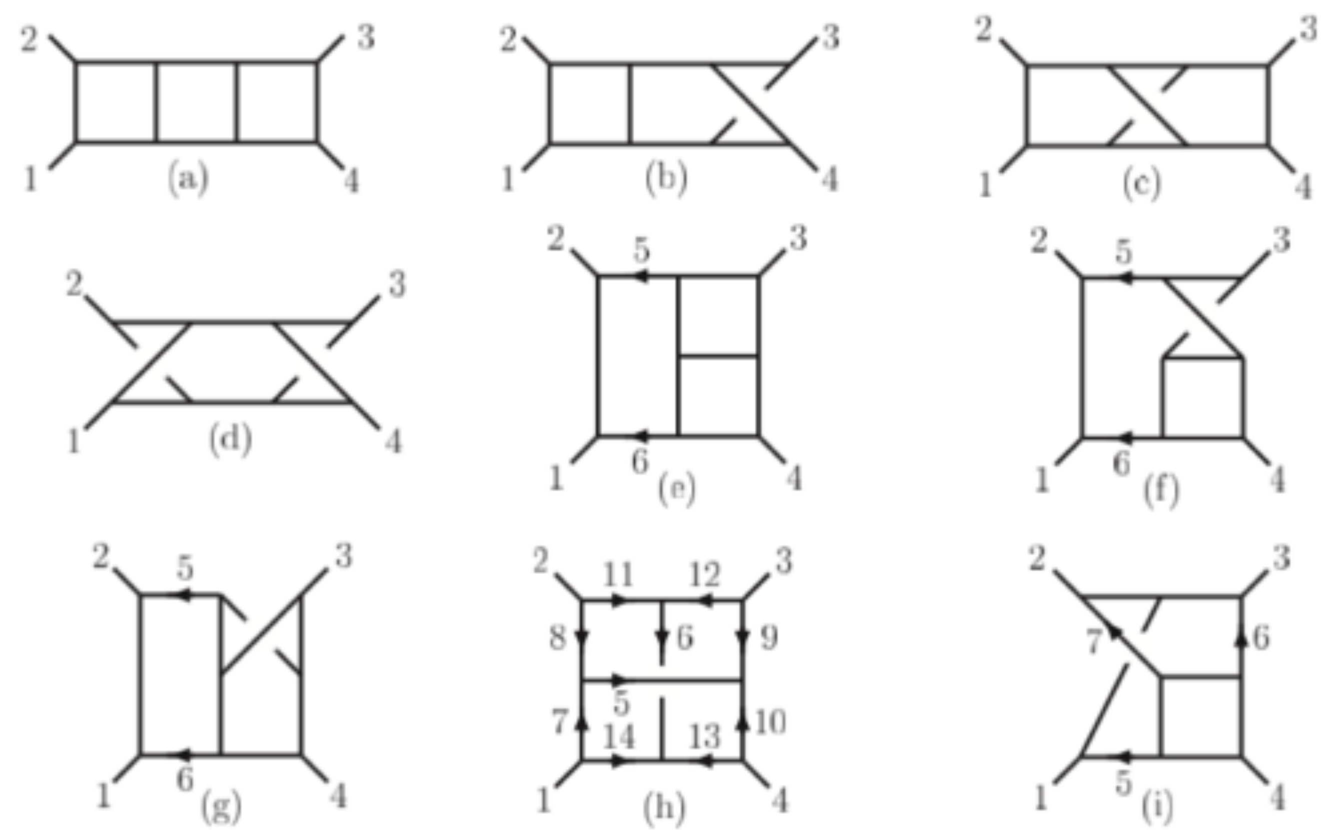
FEYNMAN INTEGRALS



$$\int$$

FEYNMAN INTEGRALS

- In general we would only require 9 Feynman Integrals
- Direct computation is very complicated: 10 propagators, divergent, …
- Remember: All Integrals are only functions of s and t (and u).
- Differential Equations!



DIFFERENTIAL EQUATIONS

- For massive Feynman integrals

$$\frac{\partial}{\partial m^2}\int d^dk\frac{1}{k^2-m^2}=\int d^dk\frac{1}{(k^2-m^2)^2}$$

But we have no masses!

- Differential operator w.r.t. Mandelstam invariants

$$\frac{\partial}{\partial s_{ij}}\mathcal{I}(s_{kl})$$

- At the integrand level we only have dependence on loop momenta and external momenta $\dfrac{1}{p^2+2k\cdot p+k^2}$

[Kotikov; Gehrmann,Remiddi]

DIFFERENTIAL EQUATIONS

- How-To: Derive a differential operator (one selected method)
  - Make an Ansatz in terms of external momenta:

$$\frac{\partial}{\partial s_{ij}} = \sum_{k,l} \alpha_{kl} \; p_k \cdot \frac{\partial}{\partial p_l}$$

  - Fix the coefficients of the Ansatz

$$\frac{\partial}{\partial s_{ij}} s_{kl} = \delta_{ij,kl}$$

- Differential operator turns Feynman integrals into other Feynman integrals

$$p_k \cdot \frac{\partial}{\partial p_l} \frac{1}{(p_l + k)^2} = -\frac{2 p_k \cdot (p_l + k)}{((p_l + k)^2)^2}$$

DIFFERENTIAL EQUATIONS+IBPS

- Differential operator turns Feynman integrals into other Feynman integrals
$$\frac{\partial}{\partial s_{12}} \mathcal{I}(s_{12}, s_{13}) = \mathcal{I}'(s_{12}, s_{13})$$

- Integration-By-Part identities: Relations among different Feynman integrals
$$\int d^d k \frac{\partial}{\partial k^\mu} \left(q^\mu f(k,p,q)\right) = 0$$

- Select "simplest" possible integrals as Master Integrals (simple: # of propagators, etc.)

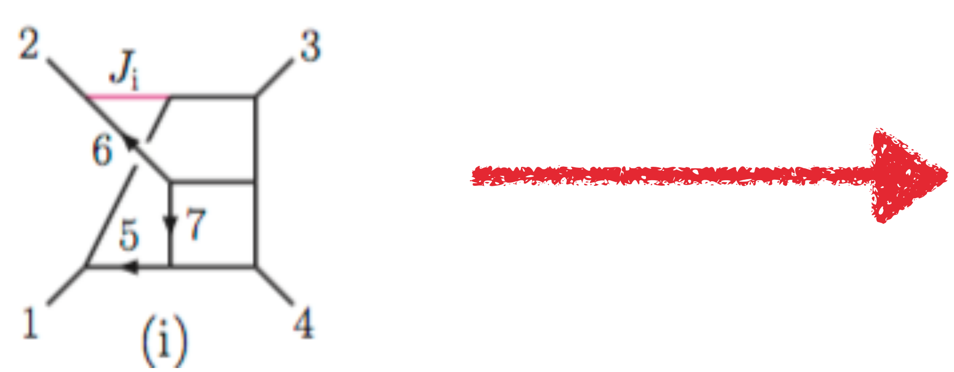
- Express every Feynman integral in terms of Master Integrals
$$\mathcal{I}'(s_{12}, s_{13}) = \sum_i c_i(s_{12}, s_{13}, \epsilon) M_i(s_{12}, s_{13})$$

DIFFERENTIAL EQUATIONS+IBPS

- System of coupled first order differential equations

$$\frac{\partial}{\partial x}\vec{M}(x) = A(x,\epsilon)\vec{M}(x) \qquad x = \frac{s_{23}}{s_{12}}$$

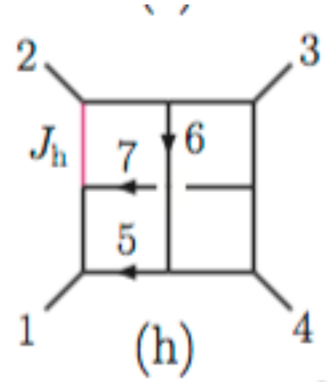
- Upside: Computing Master Integrals is now a well known problem of solving one-parameter differential equations.
- Downside: To compute 1 Integral that you are interested in you need to solve many Master Integrals.



113 Master Integrals

DIFFERENTIAL EQUATIONS+IBPS

- IBP Reduction to Master Integrals is computationally intense
- Computing 4 point 2 loop integrals is pushing the limit
- To compute



  - ~100 GB of reduction output
  - ~12.800.000 Integrals reduced
- Many public tools; We used a private code.

DIFFERENTIAL EQUATIONS+IBPS

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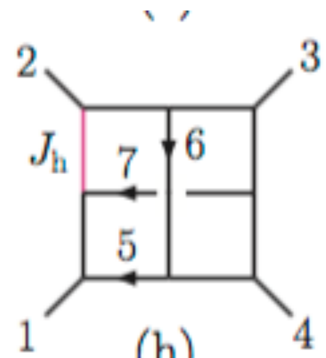

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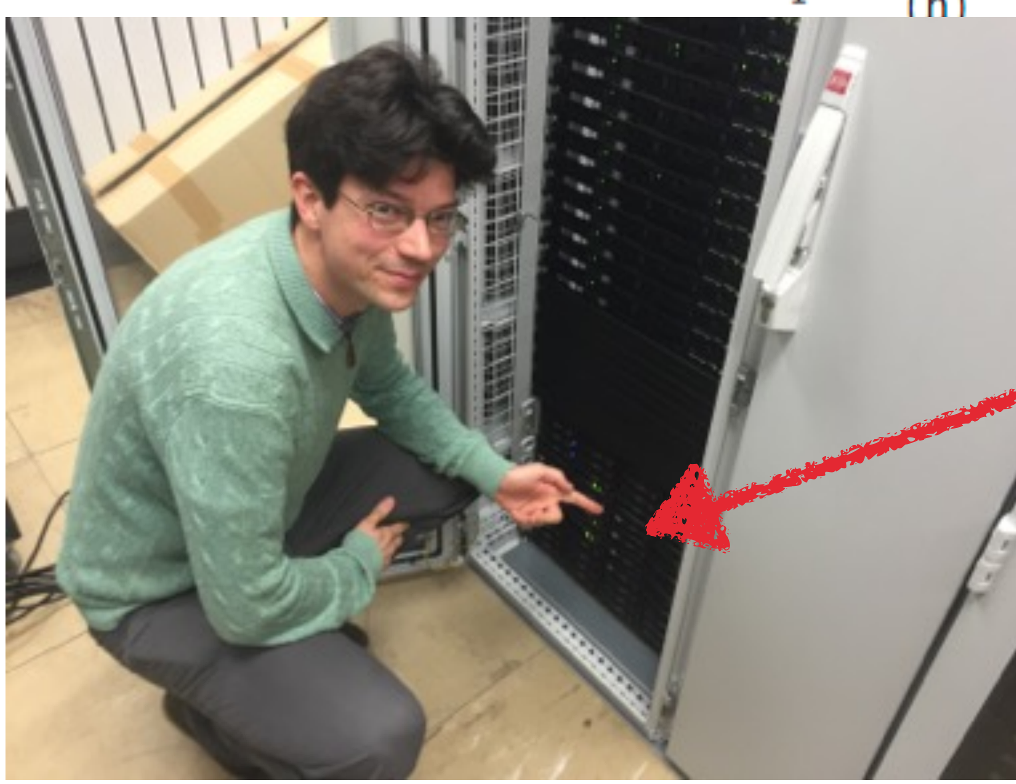


Our fourth collaborator: u0001

DIFFERENTIAL EQUATIONS+IBPS

- Structure of the differential equations

$$\frac{\partial}{\partial x}\vec{M}(x) = \left[\frac{a(x,\epsilon)}{x} + \frac{b(x,\epsilon)}{1+x}\right]\vec{M}(x)$$

Singularities only at x=0,1

- Choose a "canonical" basis: [Arkani-Hamed et al; Henn]

$$\frac{\partial}{\partial x}\vec{M}_c(x) = \epsilon\left[\frac{a_c}{x} + \frac{b_c}{1+x}\right]\vec{M}_c(x)$$

$$\vec{M} = T_c(x,\epsilon)\vec{M}_c$$

DIFFERENTIAL EQUATIONS+IBPS

- Canonical Basis $$\frac{\partial}{\partial x}\vec{M}_c(x) = \epsilon\left[\frac{a_c}{x} + \frac{b_c}{1+x}\right]\vec{M}_c(x)$$
  - Choose your Master Integrals **wisely**:
    Integrals with normalised (unit) leading singularities!

    [Arkani-Hamed et al; Henn]
  - Integrals with d-Log form in 4 dimensions
    $$M = \int \frac{d\alpha_1}{\alpha_1}\ldots\frac{d\alpha_n}{\alpha_n}$$
  - Computing d-Log form for every Master Integral can be tricky
  - Provides very natural building blocks for amplitudes

    Remember: 1 Integrand computation based
    on the fact that the entire integrand should take d-Log form!

DIFFERENTIAL EQUATIONS+IBPS

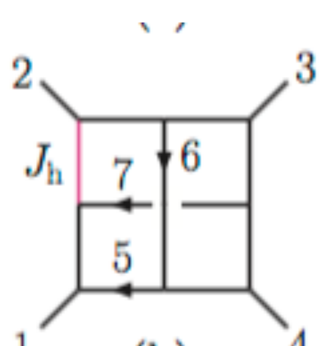
$$\vec{M} = T_c(x,\epsilon)\vec{M}_c$$

- Canonical Basis

$$\frac{\partial}{\partial x}\vec{M}_c(x) = \epsilon\left[\frac{a_c}{x} + \frac{b_c}{1+x}\right]\vec{M}_c(x)$$

  - Alternative: For rational transformations with one-parameter differential equations:
    Algorithmic solution for certain Feynman Integrals
    [Lee;Moser;Barkatou,Pfluegel]

  - Application of algorithm for large systems is computationally intense.



(h)

- Contains a 8x8 coupled sub-sector
  Took a while …
  Intermediate expression swell.

- Output contains ugly numbers …

$$\partial_x M_{76} = \left(\frac{622918401664466563147762 7\epsilon}{690270245461821024000000 0(1+x)} - \frac{248402382303128086741058389\epsilon}{759297270008003126400000000x}\right)M_1 + \dots$$

DIFFERENTIAL EQUATIONS+IBPS

- Once a canonical form is obtained:
  Solve as Laurent series in ϵ

$$\vec{M}_c(x) = \mathcal{P}e^{\epsilon\int dx\left(\frac{a_c}{x}+\frac{b_c}{1+x}\right)}\vec{M}_c(x_0)$$

$$= \left[1+\epsilon\int dx\left(\frac{a_c}{x}+\frac{b_c}{1+x}\right)+\dots\right]\vec{M}_c(x_0)$$

- Natural functions for solution:
  Harmonic Polylogarithms

$$H_{a_n,a_{n-1},\dots,a_1}(x) = \int_0^x dx' \frac{H_{a_{n-1},\dots,a_1}(x')}{x'-a_n} \qquad a_i \in \{0,-1\}$$

$$H_0(x) = log(x)$$

$$H_{0,1}(x) = -Li(x)$$

- # of integrations: **weight**
- weight of ϵ =-1

DIFFERENTIAL EQUATIONS+IBPS

$$\vec{M}_c(x) = \mathcal{P}e^{\epsilon \int dx \left(\frac{a_c}{x} + \frac{b_c}{1+x}\right)} \vec{M}_c(x_0)$$

$$= \left[1 + \epsilon \int dx \left(\frac{a_c}{x} + \frac{b_c}{1+x}\right) + \ldots\right] \vec{M}_c(x_0)$$

- 1 Integration always with one power in ϵ
  Functions of uniform (transcendental) weight
- No rational pre-factors depending on x:
  Pure Functions of uniform weight

BOUNDARY CONDITIONS

$$\vec{M}_c(x) = \mathcal{P}e^{\epsilon \int dx \left(\frac{a_c}{x} + \frac{b_c}{1+x}\right)} \vec{M}_c(x_0)$$

- Require boundary conditions for solution $\vec{M}_c(x_0)$
- By requiring consistency conditions
  and a few three loop form factor integrals we fixed all of them

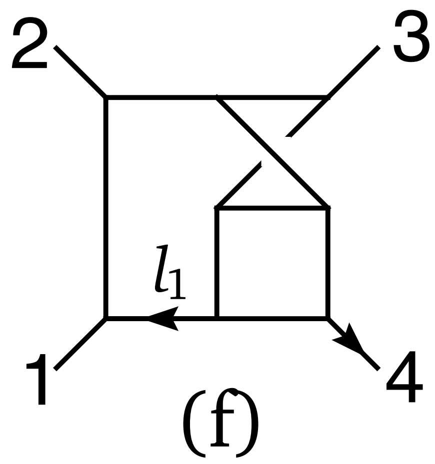
(a)

- Given in terms of Zeta values

$$\zeta(n) = Li_n(1)$$

- Uniform weight

MASTER INTEGRAL EXAMPLE

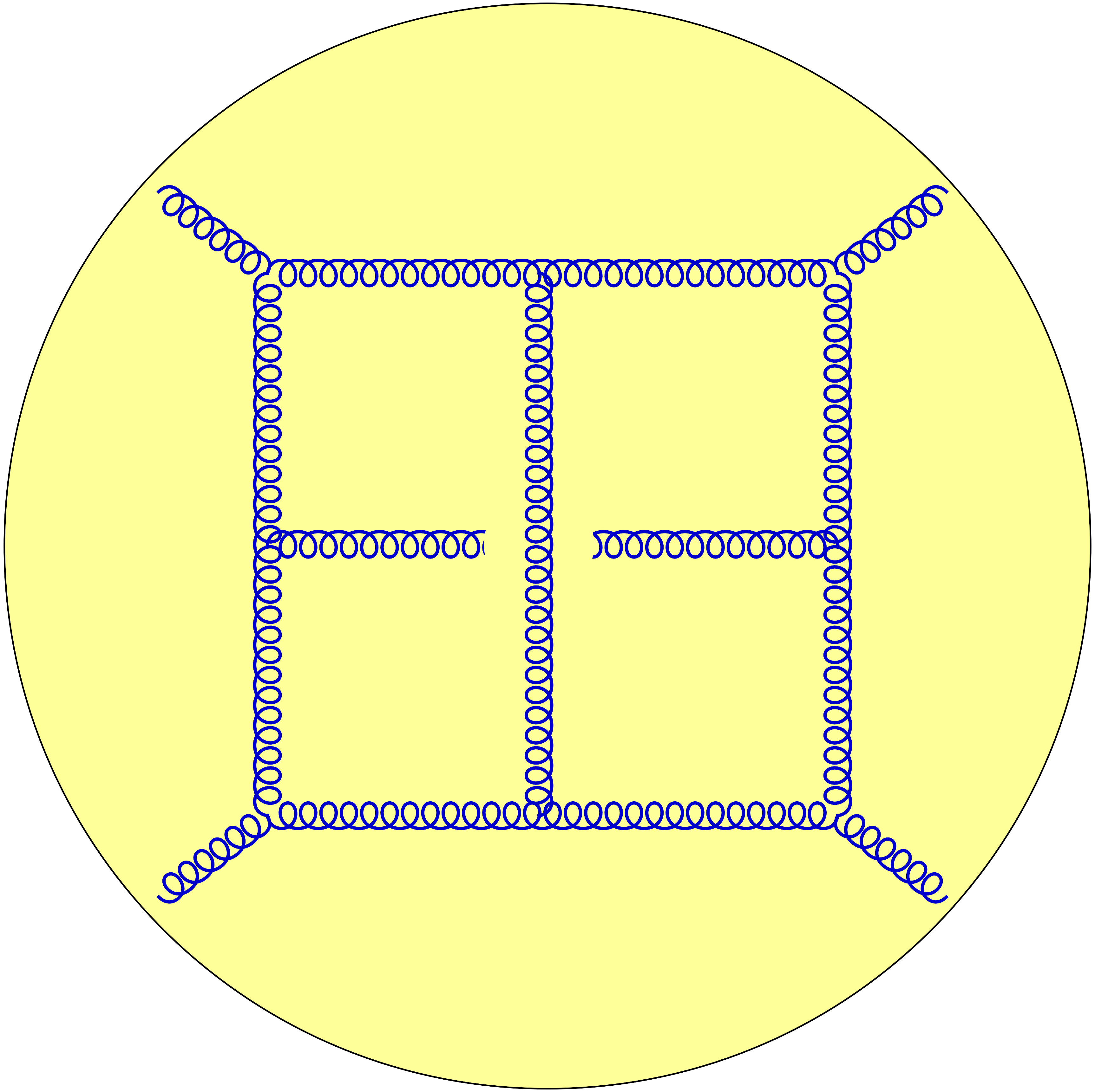


- Unit leading singularity basis:

$$I = s(s+t)I_f[(l_1+p_4)^4]$$

$$\begin{aligned} I = & -\frac{1}{\epsilon^6}\frac{47}{36} \\ & +\frac{1}{\epsilon^5}\left[-\frac{8i\pi}{3}+\frac{8H_{-1}}{3}-\frac{3H_0}{4}\right] \\ & +\frac{1}{\epsilon^4}\left[-4H_{-1,-1}+H_{-1,0}+\frac{H_{0,0}}{4}+\frac{503\zeta_2}{24}+4i\pi H_{-1}-i\pi H_0+H_{-2}\right] \\ & +\frac{1}{\epsilon^3}\left[2i\pi H_{0,0}+2H_{-2,-1}-2H_{-2,0}-2H_{-1,0,0}+\frac{21}{4}H_{0,0,0}+31i\pi\zeta_2\right. \\ & \qquad \left.+\frac{715\zeta_3}{36}-2i\pi H_{-2}-33\zeta_2 H_{-1}+\frac{355\zeta_2 H_0}{24}-2H_{-3}\right] \\ & +\mathcal{O}(\epsilon^{-2}) \end{aligned}$$

THE AMPLITUDE

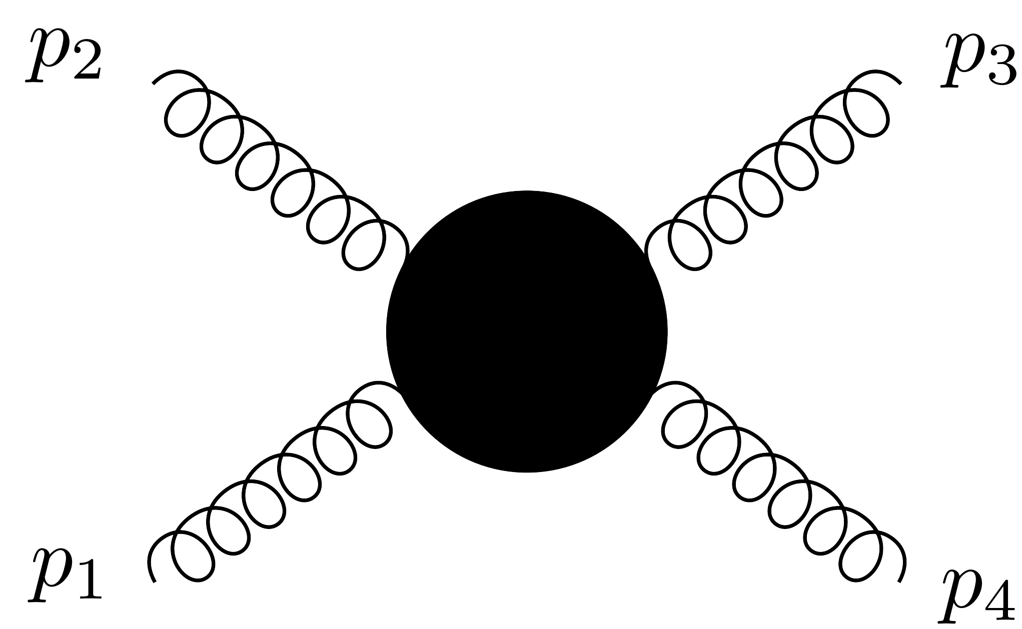


AMPLITUDE

- Computed all required Master integrals
  [Henn,BM,Smirnov, to appear]
- Checked that all available integrands for 4-point amplitude give the same result.
- Reproduced previously known planar results.
- **Result**: First four point scattering amplitude in four dimensional non-abelian gauge theory at three loops with finite n_c dependence.

What can we learn from that?

COLOUR AMPLITUDES



- 4 particle scattering:
  4 colour indices!
- Represent the amplitude
  in terms of colour-stripped amplitudes

$$\mathcal{A}(s,t) = \sum_{i=1}^{6} C_i \mathcal{A}_{(i)}(s,t)$$

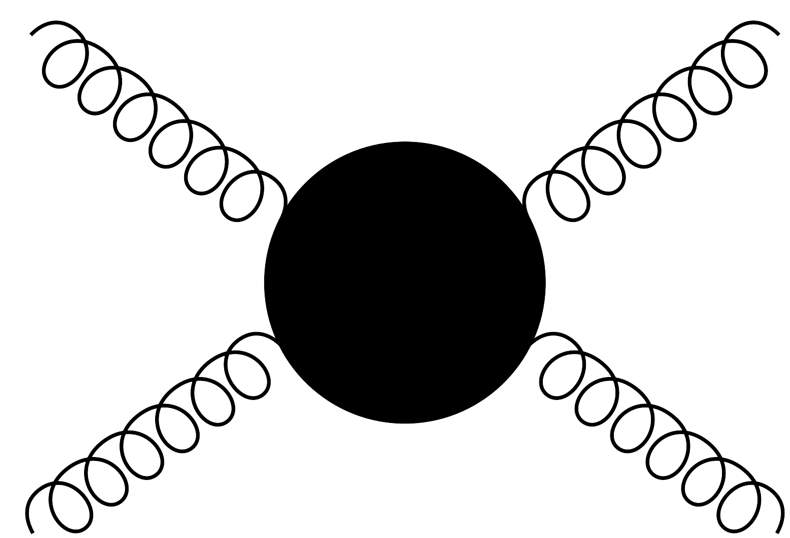
$$tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) = tr(1234).$$

$$C_1 = tr(1234) + tr(1432) \qquad C_4 = tr(12)tr(34)$$
$$C_2 = tr(1243) + tr(1342) \qquad C_5 = tr(13)tr(24)$$
$$C_3 = tr(1423) + tr(1324) \qquad C_6 = tr(14)tr(23)$$

- Generators in fundamental representation of SU(N)

COLOUR AMPLITUDES

$$\mathcal{A}(s,t) = \sum_{i=1}^{6} C_i \mathcal{A}_{(i)}(s,t)$$



- At L loops:

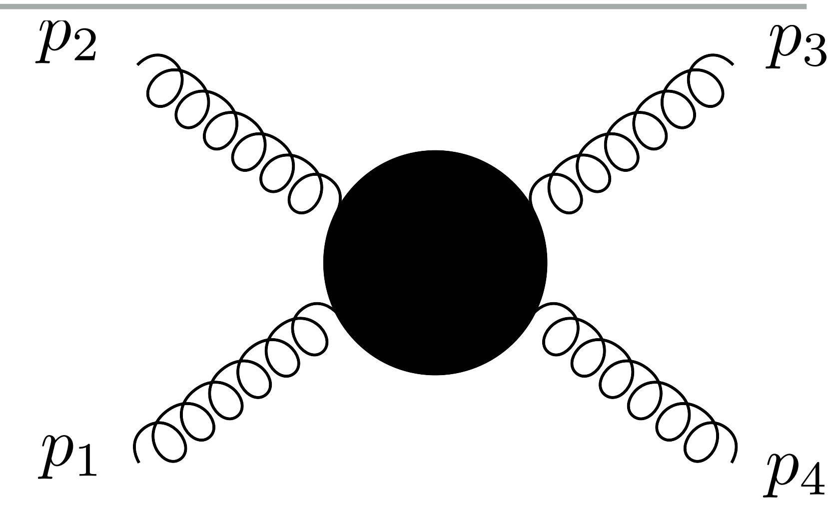
$$\mathcal{A}_{(i)} = \sum_{j=0}^{L} n_c^{L-j} \mathcal{A}_{(i)}^{(L,j)}$$

- Intricate colour identities among colour stripped amplitudes

$$\begin{aligned} 6\sum_{\lambda=1}^{3} A_\lambda^{(L,L-2)} - \sum_{\lambda=4}^{6} A_\lambda^{(L,L-1)} &= 0\,, \\ A_{\lambda+3}^{(L,L-1)} + A_\lambda^{(L,L)} &= \text{independent of } \lambda\,, \\ \sum_{\lambda=1}^{3} A_\lambda^{(L,L)} &= 0\,, \end{aligned}$$

- Kleiss-Kuijf, U(1) decoupling, etc.

[DelDuca,Dixon,Maltoni; Bern,Kosower; Kleiss Kuijf; {Naculich,Nastase,Schnitzer};...]



COLOUR AMPLITUDES

$$\mathcal{A}(s,t) = \sum_{i=1}^{6} C_i \mathcal{A}_{(i)}(s,t)$$

- Let's define a colour operator Catani-Style Action of a colour operator on a fundamental generator with adjoint index

$$\mathbf{T}_1^{a_5} T^{a_{\bar{1}}} = -i f^{a_5 a_{\bar{1}} a_6} T^{a_6} .$$

- Can act on colour traces of our amplitude.

$$\vec{A} = \begin{pmatrix} \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_6 \end{pmatrix} \qquad (\mathbf{T}_1 + \mathbf{T}_2)^2 \vec{A} = \begin{pmatrix} \frac{n_c}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{n_c}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & n_c & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & n_c & 0 \\ 0 & 0 & 1 & 0 & 0 & n_c \end{pmatrix} \vec{A}$$


100° F

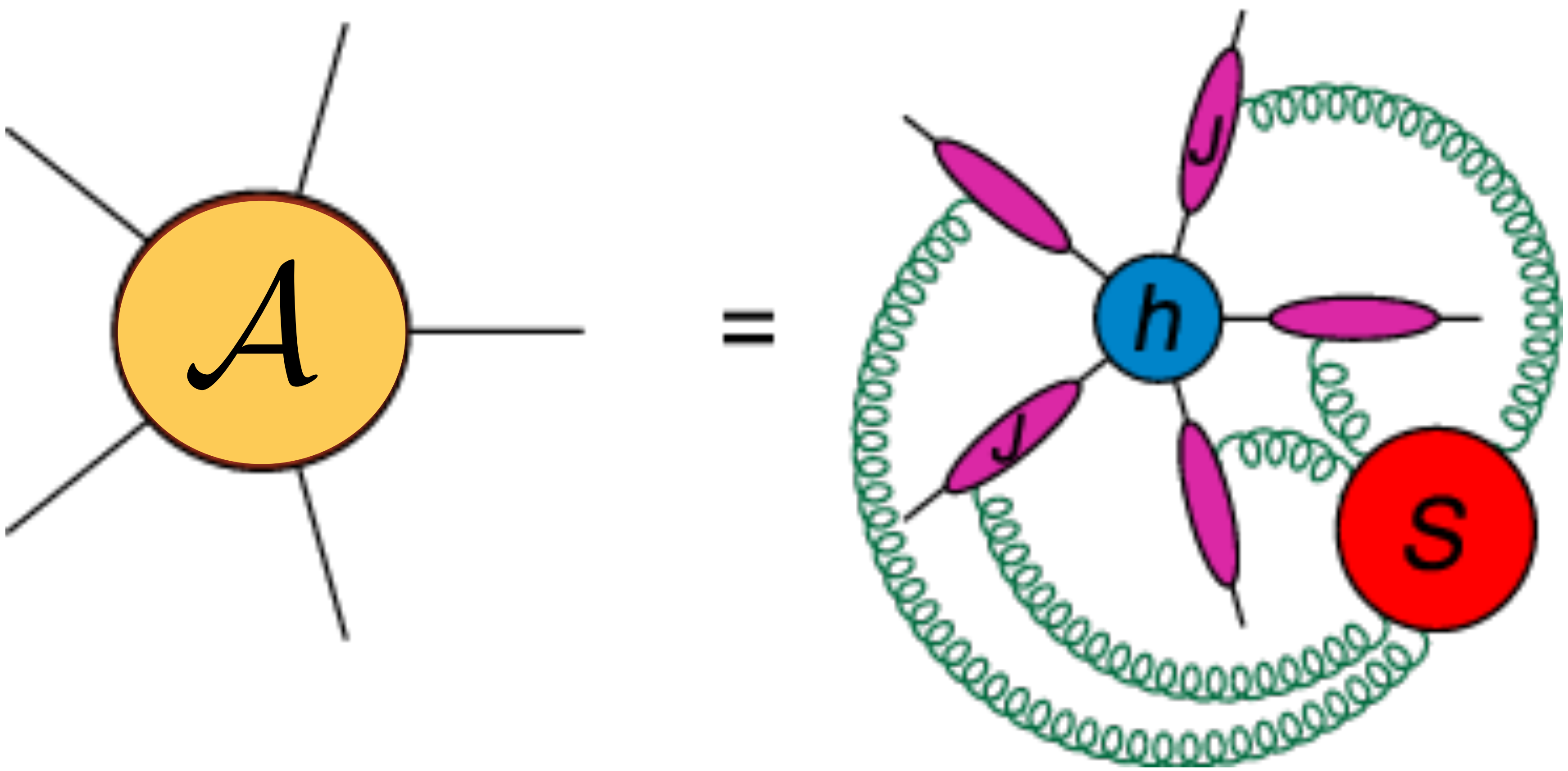
GENERAL IR STRUCTURE

- Massless gauge theory: Universal IR structure!

$$\mathcal{A}(p_i,\epsilon) = \mathbf{Z}(p_i,\epsilon)\mathcal{A}^f(p_i,\epsilon)$$

universal IR poles

IR finite



GENERAL IR STRUCTURE

- Massless gauge theory: Universal IR structure!

$$\mathcal{A}(p_i,\epsilon) = \mathbf{Z}(p_i,\epsilon)\mathcal{A}^f(p_i,\epsilon)$$

universal IR poles

IR finite

- Same structure for N=4 SYM as for QCD

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

- **Bold** letters: Composed of colour operators acting on external legs

- True for arbitrary number of loops and legs!

GENERAL IR STRUCTURE

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

- Z-factor is related to Catani's IR operator

[Catani;Teyeda-Yeomans, Sterman;Dixon,Mert Aybat,Sterman]

$\mathbf{I}_1$ $\mathbf{I}_2$

- Describes the 1 and 2 loop IR poles of any massless particle scattering amplitude.
- The Nr.1 check of your favourite amplitude computation!
- Generalisation to all loops and n partons

GENERAL IR STRUCTURE

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

- Dipole Formula and Corrections

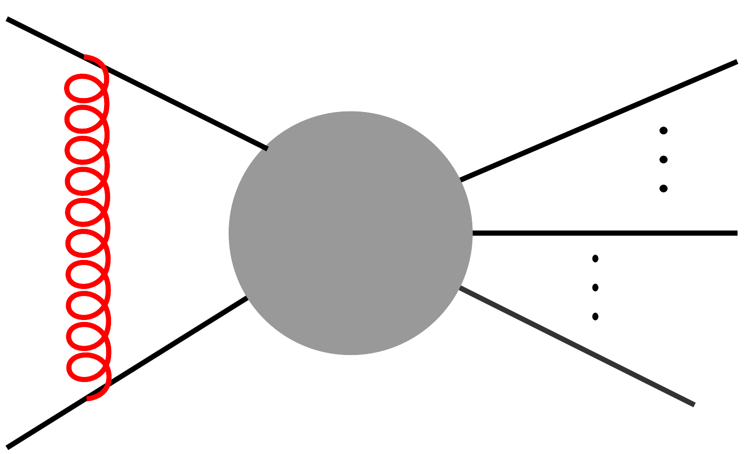
$$\mathbf{\Gamma}_n(\{p_i\},\mu^2,\alpha(\mu^2)) = \mathbf{\Gamma}_n^{\text{dip.}}(\{p_i\},\mu^2,\alpha(\mu^2)) + \mathbf{\Delta}(\{\rho_{ijkl}\}).$$

$$\mathbf{\Gamma}_n^{\text{dip.}}(\{p_i\},\mu^2,\alpha(\mu^2)) = -\frac{1}{2}\gamma_c(\alpha_S)\sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j\log\left(\frac{-s_{ij}}{\mu^2}\right) + \mathbb{I}\sum_{i=1}^{n}\gamma_J^{(i)}(\alpha).$$

- Relates only two colour charged lines: Dipole
- Theory dependence (N=4 SYM vs. QCD) in the anomalous dimensions

γ_J γ_c

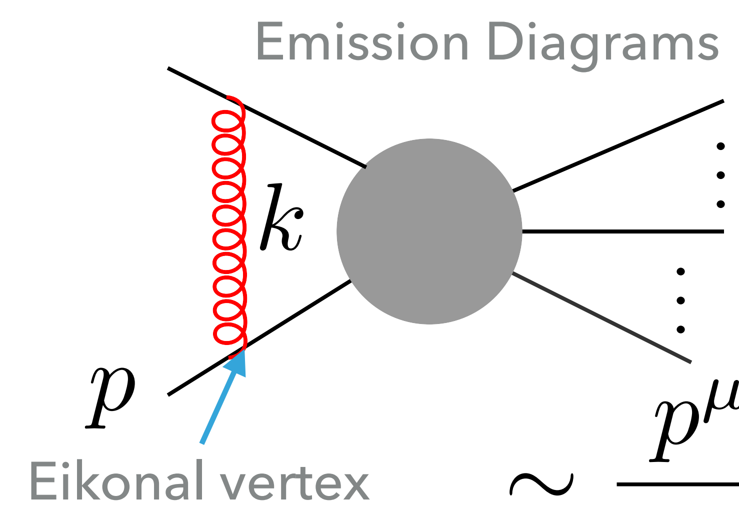
perturbative expansion



GENERAL IR STRUCTURE

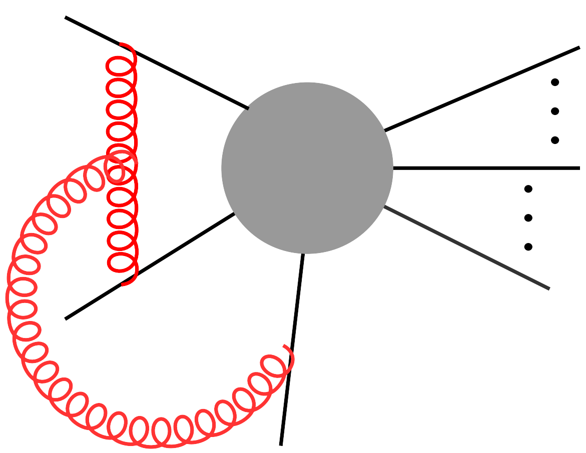
$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

- 1 loop: Only dipoles

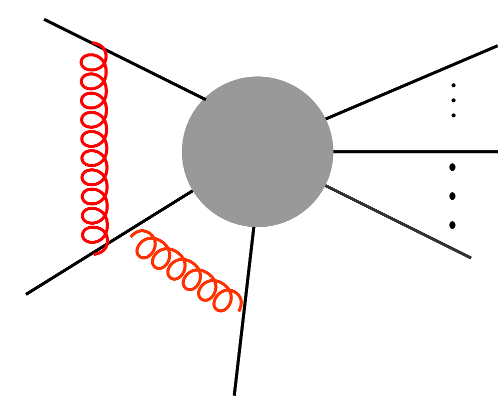


$$\sim \frac{p^\mu}{p\cdot k}$$

- Why only dipoles at 2 loop?



- Diagram vanishes :)
- Other 3 line diagrams exponentiate such that they reproduce the 1 loop colour structure



[Dixon,Mert Aybat,Sterman]

GENERAL IR STRUCTURE

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

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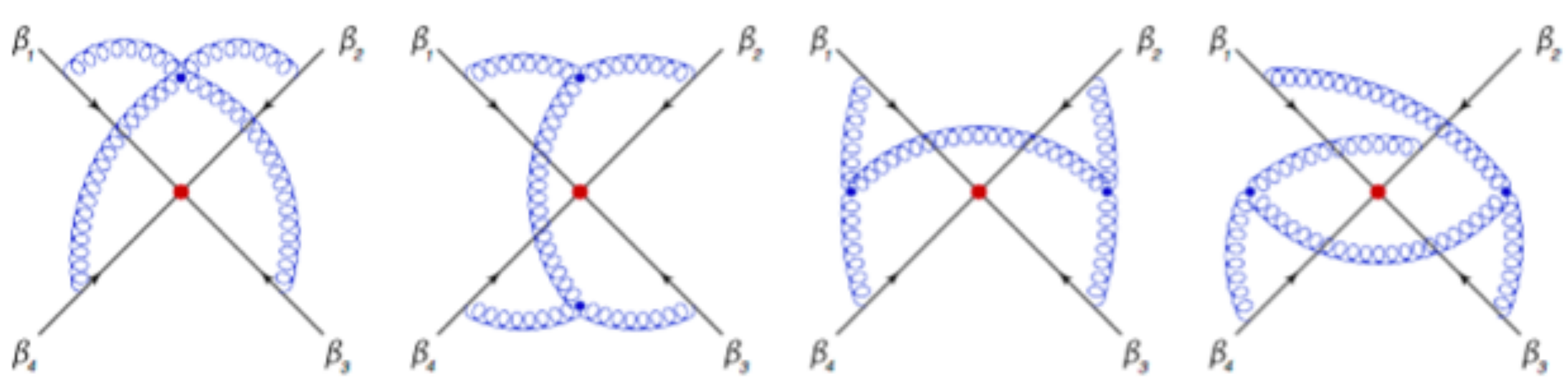
$$\mathbf{\Gamma}_n(\{p_i\},\mu^2,\alpha(\mu^2)) = \mathbf{\Gamma}_n^{\mathrm{dip.}}(\{p_i\},\mu^2,\alpha(\mu^2)) + \mathbf{\Delta}(\{\rho_{ijkl}\}).$$

- Corrections start at 3 loops!

$$\Delta_n^{(3)}(\{\rho_{ijkl}\}) = 16 f_{abe}f_{cde}\Big\{-C\sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c \quad + \tag{4.1}$$

$$\sum_{1\leq i<j<k<l\leq n}\left[\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathscr{F}(\rho_{ikjl},\rho_{iljk}) + \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathscr{F}(\rho_{ijkl},\rho_{ilkj}) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathscr{F}(\rho_{ijlk},\rho_{iklj})\right]\Big\},$$



[Almelid, Duhr, Gardi, 2015]

GENERAL IR STRUCTURE

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

- Dipole Formula and Corrections

$$\mathbf{\Gamma}_n(\{p_i\},\mu^2,\alpha(\mu^2)) = \mathbf{\Gamma}_n^{\text{dip.}}(\{p_i\},\mu^2,\alpha(\mu^2)) + \mathbf{\Delta}(\{\rho_{ijkl}\}).$$

- Corrections start at 3 loops!

$$\Delta_n^{(3)}\left(\{\rho_{ijkl}\}\right) = 16 f_{abe} f_{cde}\Big\{-C\sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c \quad + \tag{4.1}$$
$$\sum_{1\leq i<j<k<l\leq n}\left[\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathscr{F}(\rho_{ikjl},\rho_{iljk}) + \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathscr{F}(\rho_{ijkl},\rho_{ilkj}) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathscr{F}(\rho_{ijlk},\rho_{iklj})\right]\Big\},$$

[Almelid, Duhr, Gardi, 2015]

- Relating three and four coloured lines - True for arbitrarily many!

- Very specific dependence on external kinematic

GENERAL IR STRUCTURE

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

- Where do the poles come from?
  - Dependence of N=4 SYM coupling on perturbative scale

$$\alpha = C \times (\mu^2)^{-\epsilon}.$$

  - In the anomalous dimension are integrals of the type

$$C^n \int d\mu^2 (\mu^2)^{-1-n\epsilon} = C^n \frac{1}{-n\epsilon}(\mu^2)^{-n\epsilon} = \frac{\alpha^n}{-n\epsilon}$$

  - We can carry out the the integral for Z exactly

$$log\left(\mathbf{Z}\right) = \frac{1}{4}\sum_{L=1}^{\infty}\alpha^L\left[\frac{\gamma_c^{(L)}}{L^2\epsilon^2}\mathbf{D_0} - \frac{\gamma_c^{(L)}}{L\epsilon}\mathbf{D} + \frac{4}{L\epsilon}\gamma_J^{(L)}\mathbb{I} + \frac{1}{L\epsilon}\mathbf{\Delta}^{(L)}\right]$$

$$\Gamma_{\text{dip.}}$$

GENERAL IR STRUCTURE

$$\mathcal{A}(p_i,\epsilon) = \mathbf{Z}(p_i,\epsilon)\mathcal{A}^f(p_i,\epsilon)$$

$$log\left(\mathbf{Z}\right) = \frac{1}{4}\sum_{L=1}^{\infty}\alpha^L\left[\frac{\gamma_c^{(L)}}{L^2\epsilon^2}\mathbf{D_0} - \frac{\gamma_c^{(L)}}{L\epsilon}\mathbf{D} + \frac{4}{L\epsilon}\gamma_J^{(L)}\mathbb{I} + \frac{1}{L\epsilon}\mathbf{\Delta}^{(L)}\right]$$

- Compare with our amplitude!
- Highly non-trivial check of [Almelid, Duhr, Gardi, 2015]
- Correct IR structure for any massless particle amplitude up to three loops and four legs

$$\mathbf{\Delta}_4^{(3)} = 4\,f_{abe}f_{cde}\Big[\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\,\mathcal{S}(x) + \mathbf{T}_4^a\mathbf{T}_1^b\mathbf{T}_2^c\mathbf{T}_3^d\,\mathcal{S}(1/x)\Big],$$

$$\mathbf{\Delta}_3^{(3)} = -C\,f_{abe}f_{cde}\sum_{\substack{i=1\ldots4\\1\leq j<k\leq 4\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\,.$$

$$\begin{aligned}\mathcal{S}(x) = &\\ &2H_{-3,-2}+2H_{-2,-3}-2H_{-3,-1,-1}+2H_{-3,-1,0}\\ &-2H_{-2,-2,-1}+2H_{-2,-2,0}-2H_{-2,-1,-2}-H_{-1,-2,-2}\\ &-H_{-1,-1,-3}+4H_{-2,-1,-1,-1}-2H_{-2,-1,-1,0}\\ &-H_{-1,-2,-1,0}-H_{-1,-1,-2,0}+\zeta_3 H_{-1,-1}+4\zeta_3\zeta_2-\zeta_5\\ &+\zeta_2(6H_{-3}-10H_{-2,-1}+6H_{-2,0}-H_{-1,-2}-H_{-1,-1,0})\\ &+i\pi\Big[2H_{-3,-1}-2H_{-3,0}+2H_{-2,-2}-4H_{-2,-1,-1}\\ &\quad+2H_{-2,-1,0}-2H_{-2,0,0}+H_{-1,-2,0}+H_{-1,-1,0,0}\\ &\quad+\zeta(3H_{-1,-1}-4H_{-2})-\zeta_3 H_{-1}\Big]\,.\end{aligned} \quad (13)$$

$$\epsilon^0$$

- Knowing the structure of the infra-red divergences we can create a finite amplitude

$$\mathcal{A}^f(p_i,\epsilon) = \mathbf{Z}^{-1}(p_i,\epsilon)\mathcal{A}(p_i,\epsilon)$$

- Planar Limit: Z becomes colour-diagonal.
- Computation of 2-Loop planar amplitude lead to discovery of all order formula for planar 4-point scattering amplitude in N=4 SYM.

[Anastasiou, Bern, Dixon, Kosower;Bern, Dixon, Smirnov; Drummond, Henn, Korchemsky, Sokatchev]

$$\mathcal{A}_1^f = \mathcal{A}_1^{(0,0)}\exp\left\{\frac{1}{2}n_c\gamma_c \log\left(\frac{-s}{\mu^2}\right)\log\left(\frac{-t}{\mu^2}\right) - \frac{1}{2}\gamma_J\left(\log\left(\frac{-s}{\mu^2}\right)+\log\left(\frac{-t}{\mu^2}\right)\right)+C\right\}$$

- Only logarithms

- We could not find a similar structure for the non-planar contributions
- Result is still strikingly simple:
  - Uniform transcendental function
  - Weight 6 harmonic poly-logarithms and Zeta values.
  - Rational polynomials that appear in the amplitude correspond to different tree-level structures of different color stripped amplitudes.

AMPLITUDE EXAMPLE AT 2 LOOPS

$$
\begin{aligned}
\mathcal{A}_1^{f\,(2,2)} = \frac{i\mathcal{K}}{x}\Big\{ & 18\zeta_2 H_{-1,0} + 24\zeta_2 H_{0,0} - 8H_{-3,-1} + 6H_{-3,0} - 6H_{-2,-2} + 2H_{-1,-3} - 2H_{-2,-1,-1} \\
& -6H_{-2,-1,0} + 2H_{-2,0,0} - 6H_{-1,-2,-1} + 2H_{-1,-2,0} - 10H_{-1,-1,-2} + 8H_{-1,-1,-1,-1} \\
& -10H_{-1,-1,-1,0} + 4H_{-1,-1,0,0} - 2H_{-1,0,0,0} - 6\zeta_2 H_{-2} - 2\zeta_3 H_{-1} + 6H_{-4} \\
& +i\pi\Big[2H_{-2,-1} + 6H_{-2,0} + 6H_{-1,-2} - 8H_{-1,-1,-1} + 10H_{-1,-1,0} - 2H_{-1,0,0} \\
& -6H_{0,0,0} - 14H_{-1}\zeta_2 + 8H_{-3} - 6\zeta_3\Big]\Big\} \\
+ \frac{i\mathcal{K}}{1+x}\Big\{ & -36\zeta_2 H_{-1,0} - 12\zeta_2 H_{0,0} + 8H_{-3,-1} - 8H_{-3,0} + 4H_{-2,-2} - 4H_{-2,-1,-1} \\
& +4H_{-2,-1,0} + 4H_{-1,-2,-1} + 12H_{-1,-1,-2} + 12H_{-1,-1,-1,0} - 4H_{-1,-1,0,0} \\
& -4H_{-1,0,0,0} + 4H_{0,0,0,0} - 78\zeta_4 + 12\zeta_2 H_{-2} + 4\zeta_3 H_{-1} - 8H_{-4} \\
& +i\pi\Big[4H_{-2,-1} - 4H_{-2,0} - 4H_{-1,-2} - 12H_{-1,-1,0} + 8H_{0,0,0} - 4\zeta_2 H_{-1} \\
& +16\zeta_2 H_0 - 8H_{-3}\Big]\Big\}
\end{aligned}
$$

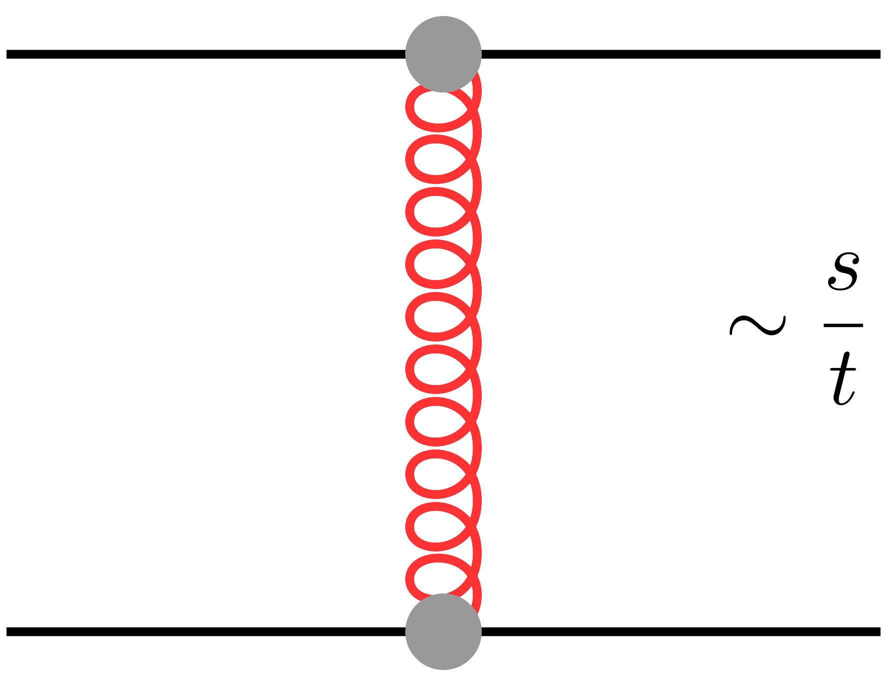
Tree level polynomials

Regge Limit

REGGE LIMIT

- Kinematic Limit:
  Center of mass energy >> momentum transfer

Regge Trajectory

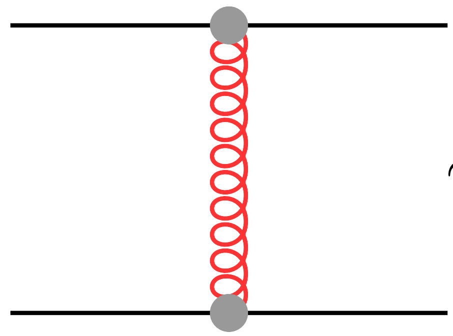


$$\sim \frac{s}{t}\left[\mathbf{T}^b f_1\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[\mathbf{T}^b f_2\right]$$

Impact Factor

helicity dependence

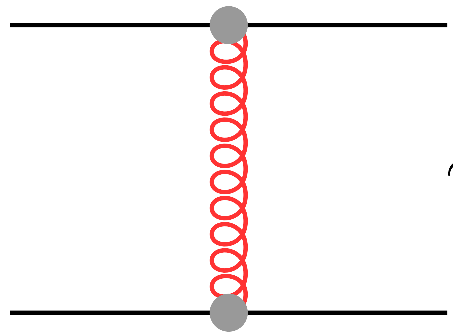
REGGE LIMIT



$$\sim \frac{s}{t}\left[\mathbf{T}^b f_1\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[\mathbf{T}^b f_2\right]$$

- Universal behavior of scattering amplitudes in high energy limit [Naculich, Schnitzer, 2007; Glover, Del Duca, 2008]
- Pole structure can be understood from infrared factorization [Del Duca,Duhr,Gard,Magnea,White]
- How is this picture violated? Universality of corrections?
  [Del Duca,Duhr,Gard,Magnea,White]
  [Del Duca, Falcioni, Magnea, Vernazza]

REGGE LIMIT



$$\sim \frac{s}{t}\left[\mathbf{T}^b f_1\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[\mathbf{T}^b f_2\right]$$

- Decompose amplitude into irreducible representations in t-channel:

$$\mathbf{8}_a \otimes \mathbf{8}_a = \mathbf{1} \oplus \mathbf{8}_a \oplus \mathbf{8}_s \oplus \mathbf{10} + \overline{\mathbf{10}} \oplus \mathbf{27} \oplus \mathbf{0}$$

- Choose color basis:

$$c_{gg}^{(1)} = \frac{1}{N_c^2-1}\,\delta^{a_4}{}_{a_1}\,\delta^{a_3}{}_{a_2}\,,$$

$$c_{gg}^{(8_s)} = \frac{N_c}{N_c^2-4}\,\frac{1}{\sqrt{N_c^2-1}}\,d^{a_1a_4b}\,d^{a_2a_3}{}_b\,,$$

$$c_{gg}^{(8_a)} = \frac{1}{N_c}\,\frac{1}{\sqrt{N_c^2-1}}\,f^{a_1a_4b}\,f^{a_2a_3}{}_b\,,$$

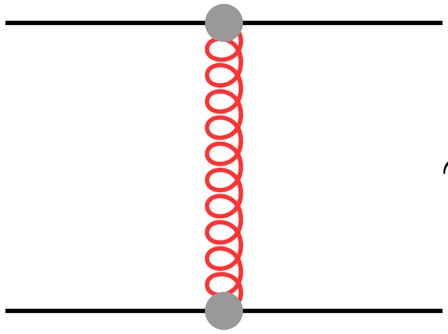
$$c_{gg}^{(10+\overline{10})} = \sqrt{\frac{2}{(N_c^2-4)(N_c^2-1)}}\left[\frac{1}{2}\left(\delta^{a_1}{}_{a_2}\,\delta^{a_3}{}_{a_4} - \delta^{a_3}{}_{a_1}\,\delta^{a_4}{}_{a_2}\right) - \frac{1}{N_c}f^{a_1a_4b}\,f^{a_2a_3}{}_b\right],$$

$$\begin{aligned} c_{gg}^{(27)} = \frac{2}{N_c\sqrt{(N_c+3)(N_c-1)}}\Bigg[&-\frac{N_c+2}{2N_c(N_c+1)}\,\delta^{a_4}{}_{a_1}\,\delta^{a_3}{}_{a_2} \\ &+\frac{N_c+2}{4N_c}\left(\delta^{a_1}{}_{a_2}\,\delta^{a_3}{}_{a_4} + \delta^{a_3}{}_{a_1}\,\delta^{a_4}{}_{a_2}\right) - \frac{N_c+4}{4(N_c+2)}\,d^{a_1a_4b}\,d^{a_2a_3}{}_b \\ &+\frac{1}{4}\left(d^{a_1a_2b}\,d^{a_3a_4}{}_b + d^{a_1a_3b}\,d^{a_2a_4}{}_b\right)\Bigg], \end{aligned}$$

$$\begin{aligned} c_{gg}^{(0)} = \frac{2}{N_c\sqrt{(N_c-3)(N_c+1)}}\Bigg[&\frac{N_c-2}{2N_c(N_c-1)}\,\delta^{a_4}{}_{a_1}\,\delta^{a_3}{}_{a_2} \\ &+\frac{N_c-2}{4N_c}\left(\delta^{a_1}{}_{a_2}\,\delta^{a_3}{}_{a_4} + \delta^{a_3}{}_{a_1}\,\delta^{a_4}{}_{a_2}\right) + \frac{N_c-4}{4(N_c-2)}\,d^{a_1a_4b}\,d^{a_2a_3}{}_b \\ &-\frac{1}{4}\left(d^{a_1a_2b}\,d^{a_3a_4}{}_b + d^{a_1a_3b}\,d^{a_2a_4}{}_b\right)\Bigg]. \end{aligned}$$

[Del Duca, Falcioni, Magnea, Vernazza]

REGGE LIMIT



$$\sim \frac{s}{t}\left[\mathbf{T}^b f_1\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[\mathbf{T}^b f_2\right]$$

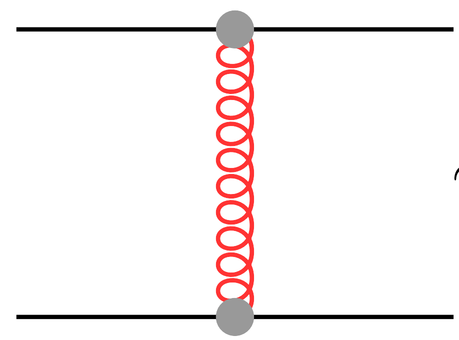
- In the octet exchange we find that we can write the amplitude in the Regge limit as

$$\mathcal{A}_{\mathbf{8}_a} \sim s^{w_{\mathbf{8}_a}} \qquad \mu^2 = -t$$

$$\begin{aligned} w_{\mathbf{8}_a}\big|_{\alpha^3} =& N_c^3\left[\frac{11\zeta_4}{48}\frac{1}{\epsilon}+\frac{5}{24}\zeta_2\zeta_3+\frac{1}{4}\zeta_5+\mathcal{O}(\epsilon)\right] \\ &+N_c\left[\frac{\zeta_2}{4}\frac{1}{\epsilon^3}-\frac{15\zeta_4}{16}\frac{1}{\epsilon}-\frac{77}{4}\zeta_2\zeta_3+\mathcal{O}(\epsilon)\right] \end{aligned}$$

- Our results also provide data for other channels

REGGE LIMIT



$$\sim \frac{s}{t}\left[\mathbf{T}^b f_1\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[\mathbf{T}^b f_2\right]$$

- Regge Limit of the IR finite amplitude can be described in terms of the tree level amplitude and Zeta values!

$$\lim_{s>>t} \mathcal{A}^f = \sum_{k,q} \alpha^k \left(\log \frac{s}{t}\right)^q \mathbf{O}_{k,q} \mathcal{A}^0 + \mathcal{O}(\epsilon)$$

$$\mathbf{S} = (\mathbf{T}_1 + \mathbf{T}_2)^2$$

$$\mathbf{T} = (\mathbf{T}_2 + \mathbf{T}_3)^2$$

$$\mathbf{O}_{2,1} = -\frac{1}{8}\zeta_3 \mathbf{T}^2\,, \tag{19}$$

$$\mathbf{O}_{3,2} = i\pi \frac{11}{24}\zeta_3 [[\mathbf{S},\mathbf{T}],\mathbf{T}]\,, \tag{20}$$

$$\begin{aligned}\mathbf{O}_{3,1} =& i\pi \frac{1}{16}\zeta_4 \left(3[\mathbf{S},\mathbf{T}]\mathbf{T} + 58[[\mathbf{S},\mathbf{T}],\mathbf{T}]\right) \\ &+ \frac{11}{6}\zeta_2\zeta_3 \left(3[\mathbf{S},\mathbf{T}]\mathbf{T} + 2[[\mathbf{S},\mathbf{T}],\mathbf{T}] - [\mathbf{S}^2,\mathbf{T}]\right) \\ &+ \left(\frac{1}{4}\zeta_5 - \frac{1}{24}\zeta_2\zeta_3\right)\mathbf{T}^3 - 4\zeta_2\zeta_3 \mathbf{T}\,.\end{aligned} \tag{21}$$

CONCLUSIONS

- We computed the first four dimensional gauge theory four particle scattering amplitude, including non-planar contributions.
- Studied the IR and Regge limit of the amplitude.
- Provides important piece of data for studying the general structure of high loop gauge theory amplitudes.
- First steps towards 4 particle scattering in realistic QFT.

Thank you!