Stefano Frixione

Collinear factorisation for e^+e^- collisions

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)
2105.06688 (SF), 2108.10261 (SF, Mattelaer, Zaro, Zhao)
2207.03265 (Bertone, Cacciari, SF, Stagnitto, Zaro, Zhao)

Zurich, 15/11/2022

Assumption:

Somewhere, someone will build an e^+e^- collider

(linear or circular)

- Cross sections stemming from e^+e^- collisions are plagued by large logs that must be resummed

- One way to do that is by means of collinear factorisation; another, with YFS

- Either way, the so-called precision tools currently available are not sufficiently accurate to meet the expected precision targets – only limited progress made since the 90's

Consider a generic cross section, sufficiently inclusive:

$$\sigma = \alpha^b \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

This is symbolic, and only useful to expose the presence of:

$$\ell = \log \frac{Q^2}{\langle E_\gamma \rangle^2}\,, \qquad L = \log \frac{Q^2}{m^2}$$

Numerology: consider the production of $Z \to ll$ at:

- $\sqrt{Q^2} = m_Z$

$$\begin{aligned} L = 24.18 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} L = 0.06 \\ 0 \le m_{ll} \le m_Z\,, \quad \ell = 8.29 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.02 \\ m_Z - 1\ \mathrm{GeV} \le m_{ll} \le m_Z\,, \quad \ell = 13.66 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.034 \end{aligned}$$

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Numerology: consider the production of $Z \to ll$ at:

- $\sqrt{Q^2} = 500$ GeV

$$\begin{aligned} L = 24.59 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} L = 0.068 \\ 0 \le m_{ll} \le m_Z\,, \quad \ell = 1.46 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.0036 \\ m_Z - 1\ \mathrm{GeV} \le m_{ll} \le m_Z\,, \quad \ell = 4.51 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.01 \end{aligned}$$

It takes a lot of brute force (i.e. fixed-order results to some $\mathcal{O}(\alpha^n)$) to overcome the enhancements due to L and ℓ.

It is always convenient to first improve by means of factorisation formulae:

$$
\begin{aligned}
d\sigma(L,\ell) &= \mathcal{K}_{soft}(\ell;L)\beta(L)d\mu && (1)\\
&= \mathcal{K}_{coll}(L;\ell)\otimes d\hat{\sigma}(\ell) && (2)
\end{aligned}
$$

Use of:

(1) YFS (resummation of ℓ)

(2) collinear factorisation (resummation of L)

Common features: $\mathcal{K}$ is an *all-order* universal factor; β and $d\hat{\sigma}$ are process-specific and computed order by order

(still brute force, but one needs comparatively less)

YFS

Aim: soft resummation for:

$$\left\{ e^+(p_1) + e^-(p_2) \longrightarrow X(p_X) + \sum_{i=0}^{n} \gamma(k_n) \right\}_{n=0}^{\infty}$$

Achieved with:

$$\begin{aligned} d\sigma(L,\ell) &= \mathcal{K}_{soft}(\ell;L)\beta(L)d\mu \\ &= e^{Y(p_1,p_2,p_X)} \sum_{n=0}^{\infty} \beta_n \left(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}p_X; \{k_i\}_{i=0}^{n}\right) d\mu_{X+n\gamma} \end{aligned}$$

This is symbolic, and stands for both the EEX and CEEX approaches [hep-ph/0006359 Jadach, Ward, Was] that build upon the original YFS work [Ann.Phys.13(61)379]

EEX: exclusive (in the photons) exponentiation, matrix element level

CEEX: coherent exclusive (in the photons) exponentiation, amplitude level, including interference

YFS

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Achieved with:

$$d\sigma(L,\ell) \quad = \quad e^{Y(p_1,p_2,p_X)} \sum_{n=0}^{\infty} \beta_n \left(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}p_X; \{k_i\}_{i=0}^{n}\right) d\mu_{X+n\gamma}$$

- Y essentially universal (process dependence only through kinematics); resums ℓ
- The soft-finite β_n are process-specific, and are constructed by means of local subtractions involving matrix elements and eikonals (i.e. *not* BN)

$$\beta_n = \alpha^b \sum_{i=0}^{n} \alpha^i \sum_{j=0}^{i} c_{n,i,j} L^j$$

- For a given n, matrix elements have different multiplicities, hence the need for the kinematic mapping $\mathcal{R}$

Collinear factorisation

Aim: collinear resummation for:

$$\left\{ k(p_k) + l(p_l) \longrightarrow X(p_X) + \sum_{i=0}^{n} a_i(k_n) \right\}_{n=0}^{\infty} \qquad a_i = e^{\pm}, \gamma \ldots$$

with initial-state particles stemming from beams:

$$(k,l) = (e^+, e^-)\,, \quad (k,l) = (e^+, \gamma)\,, \quad (k,l) = (\gamma, e^-)\,, \quad (k,l) = (\gamma, \gamma)\,, \ldots$$

Master formula:

$$\begin{aligned} d\sigma(L,\ell) &= \mathcal{K}_{coll}(L;\ell) \otimes d\hat{\sigma}(\ell) \\ \longrightarrow \quad d\sigma_{kl} &= \sum_{ij} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2)\, \Gamma_{j/l}(z_-, \mu^2, m^2) \\ &\qquad \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2; p_X, \{k_i\}_{i=0}^{n}) \end{aligned}$$

- $\Gamma_{\alpha/\beta}$ universal (the PDF); resums L
- The collinear-finite $d\hat{\sigma}_{ij}$ are process-specific, and are the standard short-distance matrix elements, constructed order by order (*with* BN). May or may not include resummation of other large logs (including ℓ)

YFS vs collinear factorisation

Both are systematically improvable in perturbation theory:
in YFS the β_n's (fixed-order), in collinear factorisation both the PDFs (logarithmic accuracy) and the $d\hat{\sigma}$'s (fixed-order, resummation)

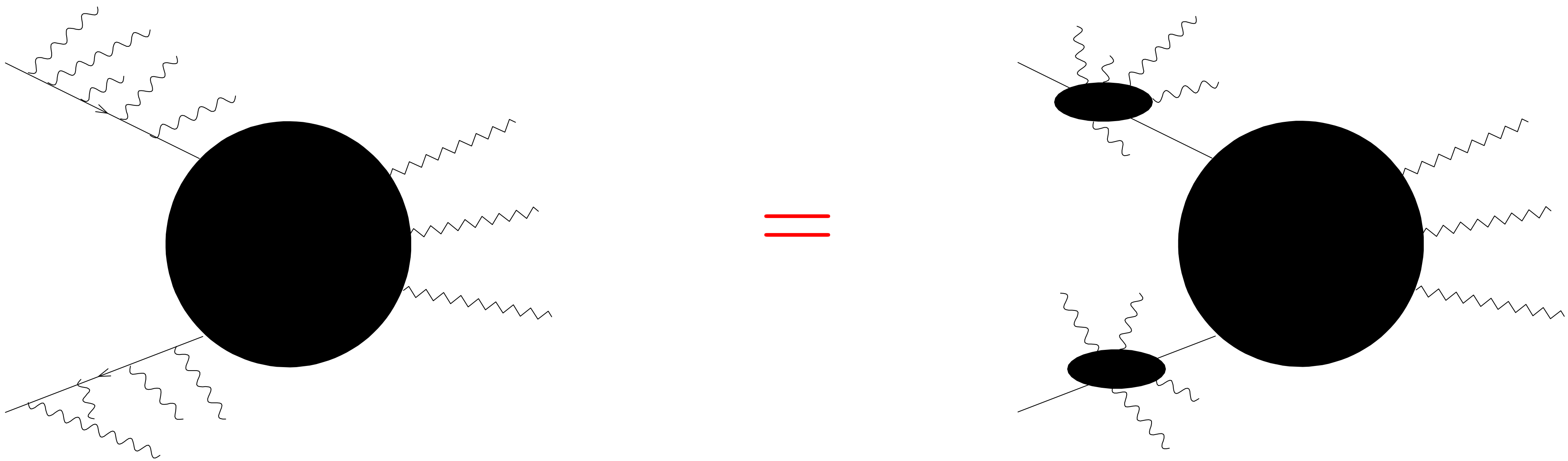
+ YFS: very little room for systematics. Exceptions are the kinematic mapping $\mathcal{R}$, and the quark masses (when the quarks are radiators). Renormalisation schemes??

− Collinear factorisation: systematic variations much larger. At the LL (used in phenomenology so far) a rigorous definition of uncertainties is impossible (parameters are arbitrary), and comparisons with YFS are largely fine tuned

− YFS: the computations of β_n are not standard (EEX) and highly non-trivial (CEEX)

+ Collinear factorisation: the computations of $d\hat{\sigma}_{ij}$ are standard

Collinear factorisation



$$d\sigma = \text{PDF} \star \text{PDF} \star d\hat{\sigma}$$

PDFs collect (universal) small-angle dynamics

Very similar to QCD*, with some notable differences:

- PDFs and power-suppressed terms can be computed perturbatively
- An object (e.g. e^-) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not

*Side benefit: we can recycle a lot of what is done for LHC

All physics simulations based on collinear factorisation done so far are based on a LL-accurate picture

This is not tenable at high energies/high statistics:

- accuracy is insufficient (see e.g. W^+W^- production)
- systematics not well defined (e.g. α is literally an arbitrary parameter)

Bear in mind:

- There is no precision physics without the ability of assessing uncertainties

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This is not tenable at high energies/high statistics:

- accuracy is insufficient (see e.g. W^+W^- production)
- systematics not well defined (e.g. α is literally an arbitrary parameter)

Step 0 was to upgrade PDFs from LL to NLL accuracy: increase of precision, and meaningful systematics, in particular factorisation-scheme dependence

z-space LO+LL PDFs $(\alpha \log(Q^2/m^2))^k$:

$\sim$ 1992

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes
- for e^-

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- matching between these two regimes
- for e^-

z-space NLO+NLL PDFs $(\alpha \log(Q^2/m^2))^k + \alpha\,(\alpha \log(Q^2/m^2))^{k-1}$:

$\longrightarrow$ 1909.03886, 1911.12040, 2105.06688 (Bertone, Cacciari, Frixione, Stagnitto)

- $0 \leq k \leq \infty$ for $z \simeq 1$
- $0 \leq k \leq 3$ for $z < 1 \iff \mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, and γ
- both numerical and analytical
- factorisation schemes: $\overline{\mathrm{MS}}$ and Δ (that has DIS-like features)

NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

Results:

$$\begin{aligned}
\Gamma_i^{[0]}(z,\mu_0^2) &= \delta_{ie^-}\delta(1-z) \\
\Gamma_{e^-}^{[1]}(z,\mu_0^2) &= \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z) \\
\Gamma_{\gamma}^{[1]}(z,\mu_0^2) &= \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z) \\
\Gamma_{e^+}^{[1]}(z,\mu_0^2) &= 0
\end{aligned}$$

Note:

- Meaningful only if $\mu_0 \sim m$
- In $\overline{\mathrm{MS}}$, $K_{ij}(z) = 0$; in general, these functions *define* a factorisation scheme

Bear in mind that PDFs are fully defined only after adopting a definite *factorisation scheme*, which is the choice of the finite terms associated with the subtraction of the collinear poles

(done by means of the $K_{ij}(z)$ functions)

- ♦ 1911.12040 $\longrightarrow$ $\overline{\mathrm{MS}}$
- ♦ 2105.06688 $\longrightarrow$ a DIS-like scheme (called Δ)

At variance with the QCD case, there is also an interesting *renormalisation-scheme* dependence of QED PDFs

(not discussed in this talk)

Definition of the Δ scheme

The idea: minimise the impact of NLO corrections

$$K_{ee}^{(\Delta)}(z) = \left[\frac{1+z^2}{1-z}\Big(2\log(1-z)+1\Big)\right]_+$$

$$K_{\gamma e}^{(\Delta)}(z) = \frac{1+(1-z)^2}{z}\Big(2\log z+1\Big)$$

Thus:

$$\Gamma_{e^-}(z,\mu_0^2) = \delta(1-z)+\frac{\alpha}{2\pi}\left[\frac{1+z^2}{1-z}\right]_+\log\frac{\mu_0^2}{m^2}$$

$$\Gamma_\gamma(z,\mu_0^2) = \frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}\log\frac{\mu_0^2}{m^2}$$

This is exactly the same as at the LO when $\mu_0 = m$

(a convenient choice for several reasons)

- The kernels $K_{ij}(z)$ also enter the evolution equations (see later)

NLL evolution (1911.12040, 2105.06688)

General idea: solve the evolution equations starting from the initial conditions computed previously

$$\frac{\partial\Gamma_i(z,\mu^2)}{\partial\log\mu^2} = \frac{\alpha(\mu)}{2\pi}\left[P_{ij}\otimes\Gamma_j\right](z,\mu^2) \quad\Longleftrightarrow\quad \frac{\partial\Gamma(z,\mu^2)}{\partial\log\mu^2} = \frac{\alpha(\mu)}{2\pi}\left[\mathbb{P}\otimes\Gamma\right](z,\mu^2)\,,$$

Two ways:

- Mellin space: suited to both numerical solution and all-order, large-z analytical solution (called *asymptotic solution*). Dominant
- Directly in z space in an integrated form: suited to fixed-order, all-z analytical solution (called *recursive solution*). Subleading

Formulae are simpler when working with a single lepton family; the physical picture is the same as that with multiple families

A technicality: owing to the running of α, it is best to evolve in t rather than in μ, with: (∼ Furmanski, Petronzio)

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi} L - \frac{\alpha^2(\mu)}{4\pi} \left(b_0 L^2 - \frac{2b_1}{b_0} L \right) + \mathcal{O}(\alpha^3) \,, \qquad L = \log \frac{\mu^2}{\mu_0^2} \,.
\end{aligned}
$$

Note:

- $t \longleftrightarrow \mu$; notation-wise, the dependence on t is equivalent to the dependence on μ
- $t = 0 \iff \mu = \mu_0$
- L is my "large log"
- Tricky: fixed-α expressions are obtained with $t = \alpha L/(2\pi)$ (and not $t = 0$)

Mellin space

Introduce the evolution operator $\mathbb{E}_N$

$$\Gamma_N(\mu^2) = \mathbb{E}_N(t)\,\Gamma_{0,N}\,, \qquad \mathbb{E}_N(0) = I\,, \qquad \Gamma_{0,N} \equiv \Gamma_N(\mu_0^2)$$

The PDFs evolution equations are then re-expressed by means of an evolution equation for the evolution operator:

$$\begin{aligned}
\frac{\partial \mathbb{E}_N^{(K)}(t)}{\partial t} &= b_0 \alpha(\mu) \mathbb{K}_N \left(I + \frac{\alpha(\mu)}{2\pi} \mathbb{K}_N \right)^{-1} \mathbb{E}_N^{(K)}(t) \\
&+ \frac{b_0 \alpha^2(\mu)}{\beta(\alpha(\mu))} \sum_{k=0}^{\infty} \left(\frac{\alpha(\mu)}{2\pi} \right)^k \\
&\qquad \times \left(I + \frac{\alpha(\mu)}{2\pi} \mathbb{K}_N \right) \mathbb{P}_N^{[k]} \left(I + \frac{\alpha(\mu)}{2\pi} \mathbb{K}_N \right)^{-1} \mathbb{E}_N^{(K)}(t)
\end{aligned}$$

- Can be solved numerically
- Can be solved analytically in a closed form under simplifying assumptions. Chiefly: large-z is equivalent to large-N

Asymptotic $\overline{\mathrm{MS}}$ solution

Non-singlet $\equiv$ singlet; photon is more complicated

$$\Gamma_{\mathrm{NLL}}(z,\mu^2) \xrightarrow{z\to 1} \frac{e^{-\gamma_{\mathrm{E}}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1}$$
$$\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\Big(L_0 - 1\Big) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \right.\right.$$
$$\left.\left. + \Big(L_0 - 1 - 2A(\xi_1)\Big) \log(1-z) - \log^2(1-z) \right] \right\}$$

where $L_0 = \log \mu_0^2/m^2$, and:

$$\begin{aligned} A(\kappa) &= -\gamma_{\mathrm{E}} - \psi_0(\kappa) \\ B(\kappa) &= \frac{1}{2}\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{12} + \gamma_{\mathrm{E}}\psi_0(\kappa) + \frac{1}{2}\psi_0(\kappa)^2 - \frac{1}{2}\psi_1(\kappa) \end{aligned}$$

with:

$$
\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\frac{20}{9}n_F + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \\
\hat{\xi}_1 &= \frac{3}{2}\,t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\,t + \mathcal{O}(\alpha t) = \lambda_0\eta_0 + \ldots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_F}{18}(3 + 4\pi^2)
\end{aligned}
$$

Remember that:

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0}\log\frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi}\,L - \frac{\alpha^2(\mu)}{4\pi}\left(b_0 L^2 - \frac{2b_1}{b_0}L\right) + \mathcal{O}(\alpha^3)\,, \qquad L = \log\frac{\mu^2}{\mu_0^2}\,.
\end{aligned}
$$

Asymptotic Δ solution

Non-singlet $\equiv$ singlet; photon is trivial

$$\Gamma_{\mathrm{NLL}}(z,\mu^2) \quad \xrightarrow{z\to 1} \quad \frac{e^{-\gamma_{\mathrm{E}}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \times \left[\left(1 + \frac{3\alpha(\mu_0)}{4\pi} L_0\right) \sum_{p=0}^{\infty} \mathcal{S}_{1,p}(z) - \frac{\alpha(\mu_0)}{\pi} L_0 \sum_{p=0}^{\infty} \mathcal{S}_{2,p}(z) \right]$$

The $\mathcal{S}_{i,p}(z)$ functions are increasingly suppressed at $z \to 1$ with growing p.
The dominant behaviour is:

$$\Gamma_{\mathrm{NLL}}(z,\mu^2) \quad \xrightarrow{z\to 1} \quad \frac{e^{-\gamma_{\mathrm{E}}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \times \left[\frac{\alpha(\mu)}{\alpha(\mu_0)} + \frac{\alpha(\mu)}{\pi} L_0 \left(A(\xi_1) + \log(1-z) + \frac{3}{4} \right) \right]$$

- A vastly different logarithmic behaviour w.r.t. the $\overline{\mathrm{MS}}$ case
  However, $\Gamma_{\mathrm{NLL}}^{(\overline{\mathrm{MS}})} - \Gamma_{\mathrm{NLL}}^{(\Delta)} = \mathcal{O}(\alpha^2)$

Key facts

- Both $\overline{\mathrm{MS}}$ and Δ results feature an integrable singularity at $z \longrightarrow 1$, basically identical to the LL one

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- In addition to that, in $\overline{\mathrm{MS}}$ there are single and double logarithmic terms

- Owing to the integrable singularity, it is essential to have large-z analytical results: the PDFs convoluted with cross sections are obtained by matching the small- and intermediate-z numerical solution with the large-z analytical one

  Analytical recursive solutions are used as cross-checks

A look at the photon:

$$\begin{aligned}
\Gamma_\gamma^{(\overline{\mathrm{MS}})}(z,\mu^2) &\xrightarrow{z\to 1} \frac{t\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{3}{2\pi\xi_1}\,\log(1-z) - \frac{t\alpha(\mu_0)^3}{\alpha(\mu)}\,\frac{1}{2\pi^2\xi_1}\,\log^3(1-z)\\
\Gamma_\gamma^{(\Delta)}(z,\mu^2) &\xrightarrow{z\to 1} \frac{1}{2\pi}\,\frac{\alpha^2(\mu_0)}{\alpha(\mu)}\,\frac{1+(1-z)^2}{z}\,L_0 + \frac{1}{2\pi\xi_1}\,\frac{t\,\alpha^2(\mu_0)}{\alpha(\mu)}\,L_0\\
&\qquad -\frac{t\,\alpha(\mu)}{2\pi\xi_1}\,\frac{e^{-\gamma_{\mathrm{E}}\xi_1}e^{\hat{\xi}_1}}{\Gamma\left(1+\xi_1\right)}\,(1-z)^{\xi_1}\,L_0\,.
\end{aligned}$$

- $\overline{\mathrm{MS}}$ vs Δ exhibits the same pattern as for (non-)singlet: logarithmic terms dominate at $z \to 1$ in $\overline{\mathrm{MS}}$, but are absent in Δ

All physics simulations based on collinear factorisation done so far are based on a LL-accurate picture

This is not tenable at high energies/high statistics:

- accuracy is insufficient (see e.g. W^+W^- production)
- systematics not well defined

Step 0 was to upgrade PDFs from LL to NLL accuracy

Step 1 (2207.03265, Bertone, Cacciari, Frixione, Stagnitto, Zaro, Zhao) is to include in the NLL PDFs thus obtained the ingredients necessary for sensible phenomenology, in particular:

- evolution with all fermion families (leptons and quarks), including their respective mass thresholds
- renormalisation schemes: $\overline{\rm MS}$, $\alpha(m_Z)$, and G_μ
- assess implications by studying realistic observables in physical processes

Sample results for:

$$\begin{aligned} e^+e^- &\longrightarrow q\bar{q} \\ e^+e^- &\longrightarrow t\bar{t} \\ e^+e^- &\longrightarrow W^+W^- \end{aligned}$$

with $q\bar{q}$ production (massless quarks) restricted to ISR QED radiation.
The other two are in the SM

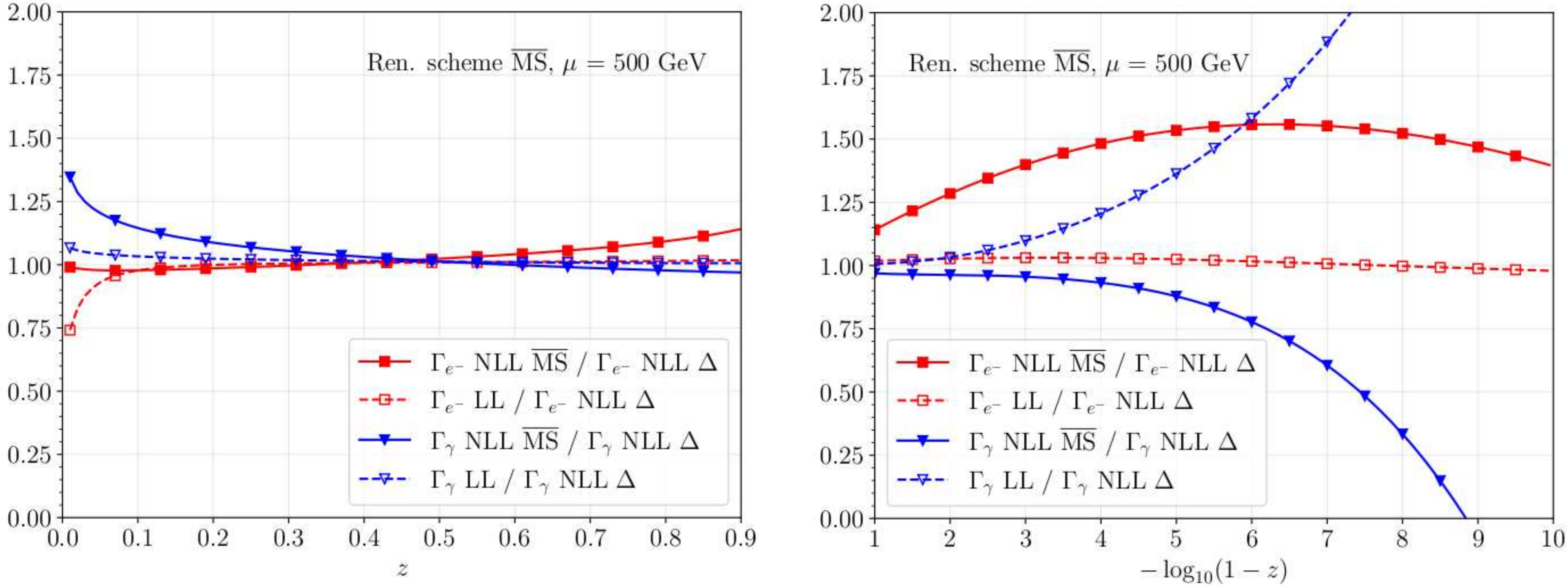
NLO accuracy, automated generation with MG5_aMC@NLO
(this version now public, but not yet as v3.X)

What is plotted:

$$\sigma(\tau_{min}) = \int d\sigma\, \Theta\left(\tau_{min} \le \frac{M^2_{p\bar{p}}}{s}\right), \qquad p = q\,, t\,, W^+$$

$\tau_{min} \sim 1$ is sensitive to soft emissions (not resummed)

Dependence of PDFs on factorisation scheme

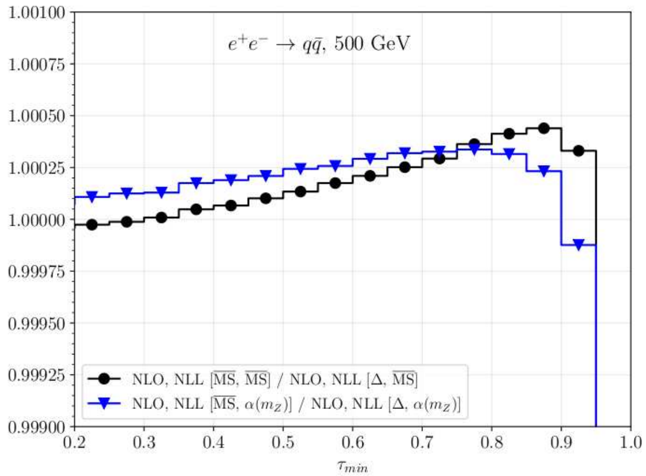


$z < 1$ $\qquad\qquad$ $z \simeq 1$

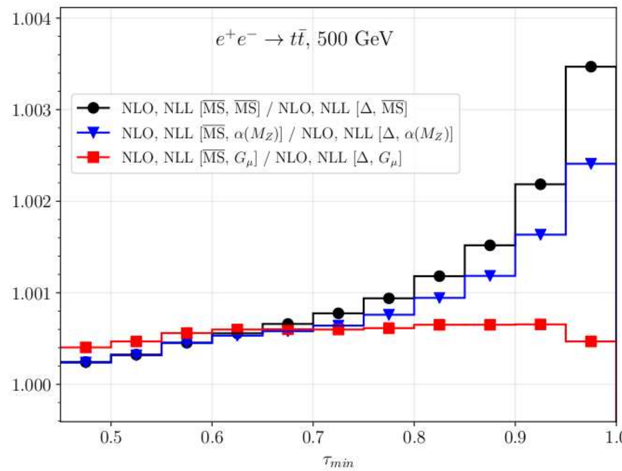
Very large dependence at the NLL at $z \to 1$ ($\mathcal{O}(1)$); this is particularly significant *(but unphysical!)* since the electron has an integrable divergence there

Electron at NLL in the Delta scheme close to the LL result (differences of $\mathcal{O}(5\%)$)

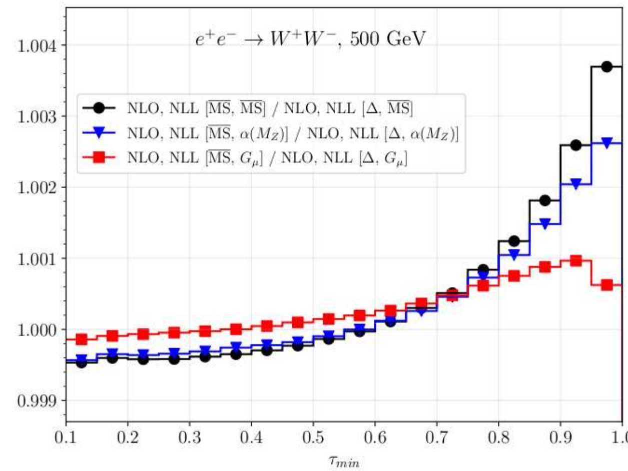
Dependence of observables on factorisation scheme



$q\bar{q}$



$t\bar{t}$



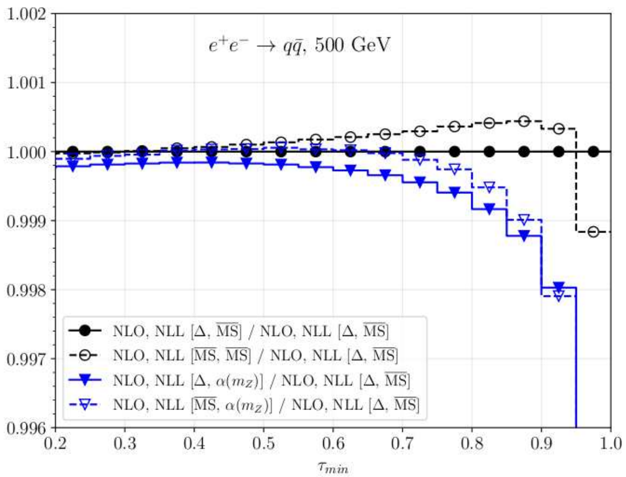
W^+W^-

$\mathcal{O}(1)$ differences for PDFs down to $\mathcal{O}(10^{-4} - 10^{-3})$ for observables

In the $\overline{\mathrm{MS}}$ scheme, huge cancellations between PDFs and short-distance cross sections

Behaviour qualitatively similar for different renormalisation schemes

Factorisation vs renormalisation scheme dependence



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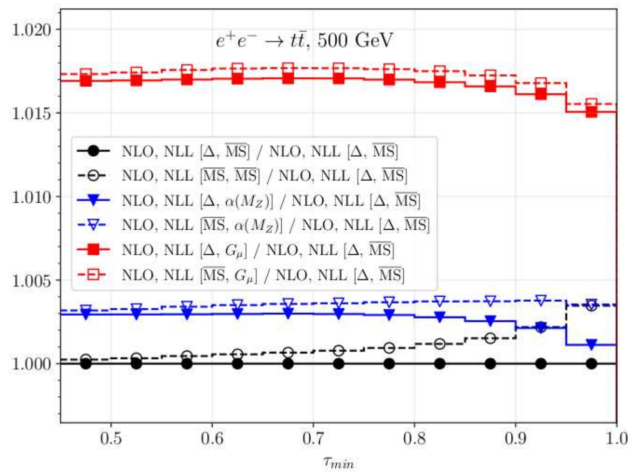

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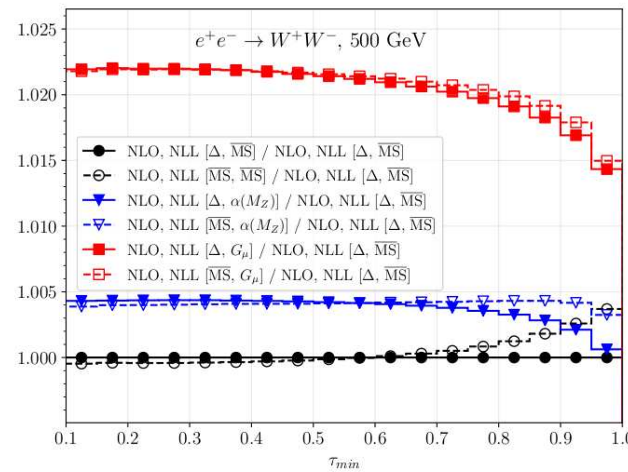
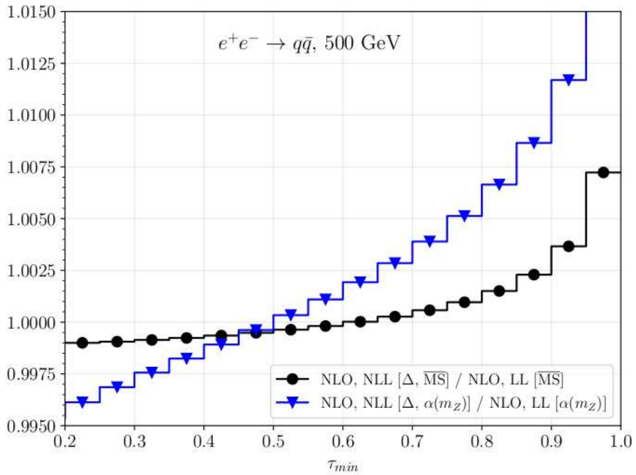


W^+W^-

Renormalisation-scheme dependence much larger than factorisation-scheme dependence, with process-dependent pattern

Depending on the precision, renormalisation scheme is an informed choice; factorisation scheme always induces a systematic

NLL vs LL



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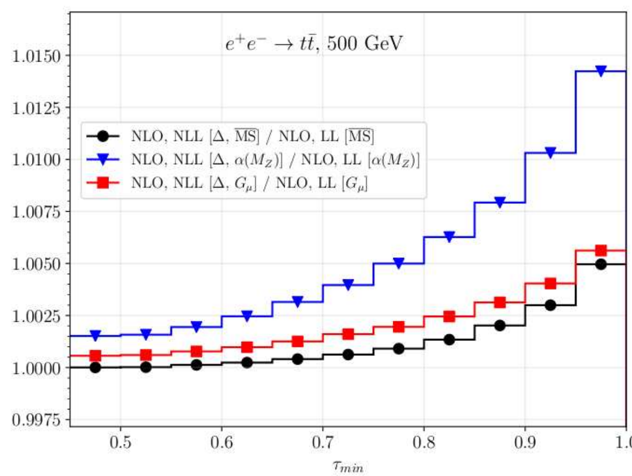

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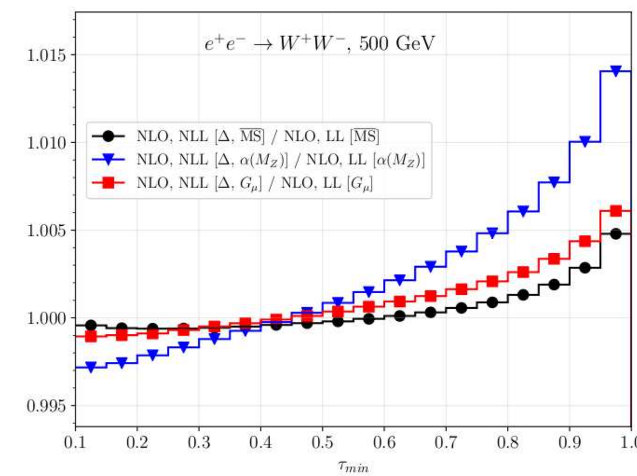
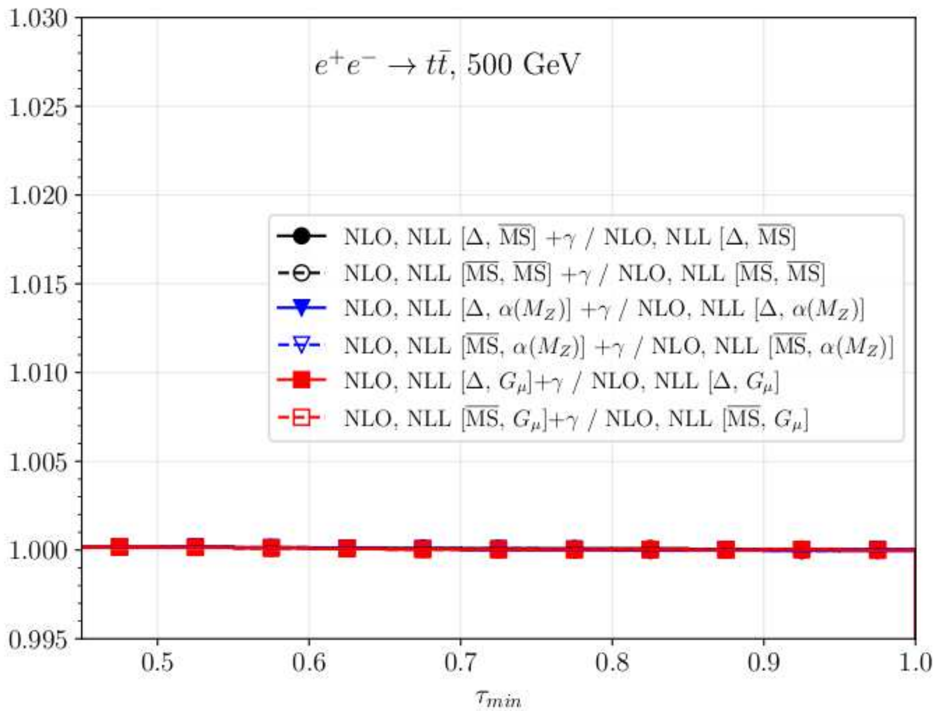


W^+W^-

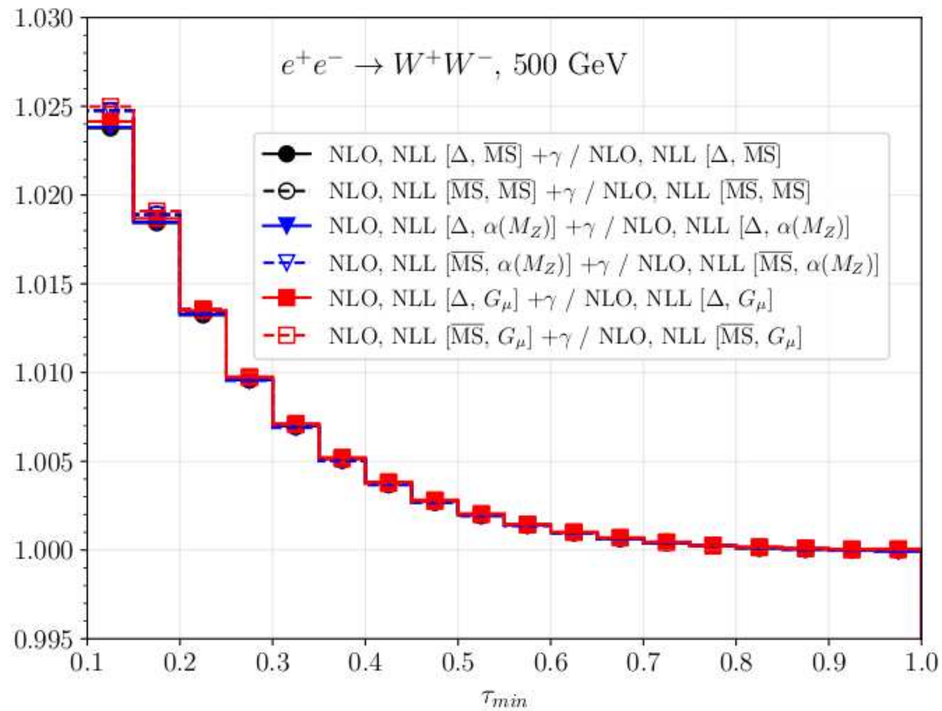
Effects are non trivial

Pattern dependent on the process (and on the observable) as well as on the renormalisation scheme

Impact of $\gamma\gamma$ channel



$t\bar{t}$



W^+W^-

Essentially independent of factorisation and renormalisation schemes: a genuine physical effect

Utterly negligible for $t\bar{t}$, significant for W^+W^- – process dependence is not surprising

Thus:

- The inclusion of NLL contributions into the electron PDF has an impact of $\mathcal{O}(1\%)$ (precise figures are observable and renormalisation-scheme dependent)
- This estimate does not include the effects of the photon PDF
- The comparison between $\overline{\mathrm{MS}}$- and Δ-based results shows differences compatible with non-zero $\mathcal{O}(\alpha^2)$ effects, as expected
- Renormalisation-scheme dependence is of $\mathcal{O}(0.5\%)$

If the target is a $10^{-\text{some large number}}$ relative precision, these effects must be taken into account

Outlook

Increasing the precision of theoretical results will be essential for the success of the physics programs at future e^+e^- colliders

QED collinear factorisation is very similar to its QCD counterpart: it is possible to recycle many of the techniques invented for the LHC

QED PDFs are now NLL accurate. It is important that they be used in the context of realistic simulations

LEP-era YFS-based results are not up to the task. What to do is in principle clear; in practice, it looks painful and of limited scope
(count the number of applications)

EXTRA SLIDES

z space

Use integrated PDFs (so as to simplify the treatment of endpoints)

$$\mathcal{F}(z,t) = \int_0^1 dy\, \Theta(y-z)\, \Gamma(y,\mu^2) \quad \Longrightarrow \quad \Gamma(z,\mu^2) = -\frac{\partial}{\partial z}\mathcal{F}(z,t)$$

in terms of which the formal solution of the evolution equation is:

$$\mathcal{F}(z,t) = \mathcal{F}(z,0) + \int_0^t du\, \frac{b_0 \alpha^2(u)}{\beta(\alpha(u))} \left[\mathbb{P}\, \overline{\otimes}\, \mathcal{F}\right](z,u)$$

By inserting the representation:

$$\mathcal{F}(z,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathcal{J}_k^{\mathrm{LL}}(z) + \frac{\alpha(t)}{2\pi}\, \mathcal{J}_k^{\mathrm{NLL}}(z) \right)$$

on both sides of the solution, one obtains recursive equations, whereby a $\mathcal{J}_k$ is determined by all $\mathcal{J}_p$ with $p < k$. The recursion starts from $\mathcal{J}_0$, which are the integrated initial conditions

For the record, the recursive equations are:

$$
\begin{aligned}
\mathcal{J}_k^{\mathrm{LL}} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\mathrm{LL}} \\
\mathcal{J}_k^{\mathrm{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{F}^{[1]}(\mu_0^2) \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \Bigg(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \\
&\qquad\qquad - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \Bigg)
\end{aligned}
$$

We have computed these for $k \leq 3$ ($\mathcal{J}^{\mathrm{LL}}$) and $k \leq 2$ ($\mathcal{J}^{\mathrm{NLL}}$), ie to $\mathcal{O}(\alpha^3)$

Results in 1911.12040 and its ancillary files

Large-z singlet and photon

As for the non-singlet, start from the asymptotic AP kernel expressions:

$$\mathbb{P}_{\mathrm{S},N} \quad \xrightarrow{N\to\infty} \quad \begin{pmatrix} -2\log\bar{N} + 2\lambda_0 & 0 \\ 0 & -\frac{2}{3}\,n_F \end{pmatrix}$$
$$+ \frac{\alpha}{2\pi} \begin{pmatrix} \frac{20}{9} n_F \log\bar{N} + \lambda_1 & 0 \\ 0 & -n_F \end{pmatrix} + \mathcal{O}(1/N) + \mathcal{O}(\alpha^2)$$

This implies

$$\begin{aligned} (\mathbb{E}_N)_{SS} &= E_N \\ M^{-1}\left[\, (\mathbb{E}_N)_{\gamma\gamma} \,\right] &= \frac{\alpha(\mu_0)}{\alpha(\mu)}\,\delta(1-z) \end{aligned}$$

$\Rightarrow$ Singlet $\equiv$ non-singlet

Photon $\equiv$ initial condition + $\alpha(0)$ scheme

Photon $\equiv$ initial condition + $\alpha(0)$ scheme $\Longrightarrow$

$$\Gamma_\gamma(z,\mu^2) = \frac{1}{2\pi}\,\frac{\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) .$$

Or: $\sim$ Weizsaecker-Williams function, plus the natural emergence of a small scale in the argument of α

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$\longrightarrow$ $1/N$ suppression of off-diagonal terms in the evolution operator is over-compensated by the δ-like peak of the electron initial-condition

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By solving the 2×2 system e.g. in $\overline{\mathrm{MS}}$:

$$\Gamma_\gamma^{(\overline{\mathrm{MS}})}(z,\mu^2) \quad \stackrel{z\to 1}{\longrightarrow} \quad \frac{t\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{3}{2\pi\xi_1}\,\log(1-z) - \frac{t\alpha(\mu_0)^3}{\alpha(\mu)}\,\frac{1}{2\pi^2\xi_1}\,\log^3(1-z)$$

A remarkable fact

Our asymptotic solutions, expanded in α, feature *all* of the terms:

$$\frac{\log^q(1-z)}{1-z} \qquad \mathrm{singlet,\ non-singlet}$$

$$\log^q(1-z) \qquad \mathrm{photon}$$

of our recursive solutions

Non-trivial; stems from keeping subleading terms (at $z \to 1$) in the AP kernels