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A sandpile model for the distribution of rainfall?

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Abstract

Recently, Peters et al. (Phys. Rev. Lett. 88 (2002) 018701)) have found power-law behaviour in the distribution of rain events. Here it is shown that the observed self-similar features in both the reservoir level and the distribution of rain events, can quantitatively be reproduced from an extension to two dimensions of the Oslo-model developed to describe the dynamics of a pile of rice. Furthermore, it is argued that the sandpile model may be able to reproduce more detailed features such as the shape of cumulus clouds. In addition, many other systems, which show self-similarity in their time variations with the same quantitative characterizations, are discussed.

Keywords: Self-organized criticality; Oslo-model; Rainfall; Hurst-exponent

1. Introduction

The occurrence of power laws in nature is usually limited to a small number of decades, which in the past has lead to some discussion whether they present a useful concept at all to describe nature [1]. Recently, Peters et al. [2] have presented power-law behaviour in the distribution of rainfall over at least four decades. In their analysis of reservoir levels, they similarly found power-law behaviour over four decades. Thus, together with the classical analysis of the water level of the Nile [3], a water reservoir experimentally shows the same exponent over seven to eight decades (on time scales from minutes to millennia). Thus, power-law behaviour in nature can indeed be observed on many scales experimentally.

In their article, Peters et al. [2] speculate that such power-law behaviour indicates that rainfall arises due to the self-organized criticality (SOC) [4] of the system of

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the atmosphere. This statement however, is not supported with results from models known to exhibit SOC. Many times in models and experiments of SOC, exponents characterizing the avalanche size-distribution function are the only quantities studied. If one wants to make claims about rainfall distribution, of course this is not sufficient, since arguably the system is much more complicated and cannot be reduced to a single number. However, as will be shown below, there are other parameters in the discussion on rainfall that can be modelled using simple cellular automata, putting stronger constraints on a possible candidate model. Among these are the water reservoir level, as discussed in Ref. [2] or the shape of clouds as discussed below. Here it is argued that a very simple sandpile model [5], when re-interpreted for the situation of the rainfall, quantitatively reproduces both the Hurst-exponent of the reservoir level, and the probability distribution exponent for rain events. Furthermore, there is evidence that the structure of some clouds may also be reproduced from the active clusters of such a model.

In the following, the main lines of the analysis of the rain experiment by Peters et al. [2] are recapitulated, in order to introduce the relevant parameters. Then the Oslo-model is briefly described, both in its original context, as well as in the interpretation used later on in the discussion of the rainfall data. In this context, I extend the model to two dimensions. The results of the 2d-model will in the end be used in the comparison with the experimental results. Finally, the results of simulations of this 2d Oslo-model are analysed in terms of a reservoir level, similarly to what was done for the rainfall data [2], as well as in terms of the avalanche size distribution. The results are then also put into a wider perspective comparing them to experiments in other systems showing similar behaviour.

2. Analysis of water levels and rain events

Rainfall has been studied for centuries [1] and power-law behaviour was also searched for in the past [6]. However, up to the analysis of Peters et al. [2], results were inconclusive and power-law behaviour was only found in a limited number of cases and over very few decades.

By defining the concept of a ‘rain event’, which is the amount of rain that falls in a period where the rain rate is non-zero, they have now found power-law behaviour in the size distribution of rain events over four decades [2]. It should be noted here that the concept of a rain event is closely related to that of an avalanche in sandpile models and in SOC in general.

In addition, the classical analysis of Hurst on the water levels of the Nile [3] has been applied to the rainfall data obtained using Doppler Radar measurements over the Baltic Sea [7]. For this, a virtual water reservoir, $X(t,\tau)$, is defined, given the rain rate

[1] The first fecords of measured rainfall date from 1441, when king Sejong of Korea invented the first rain gauge and started collecting and recording scientific rainfall measurements; Information from the Korea Meteorological Administration (KMA).

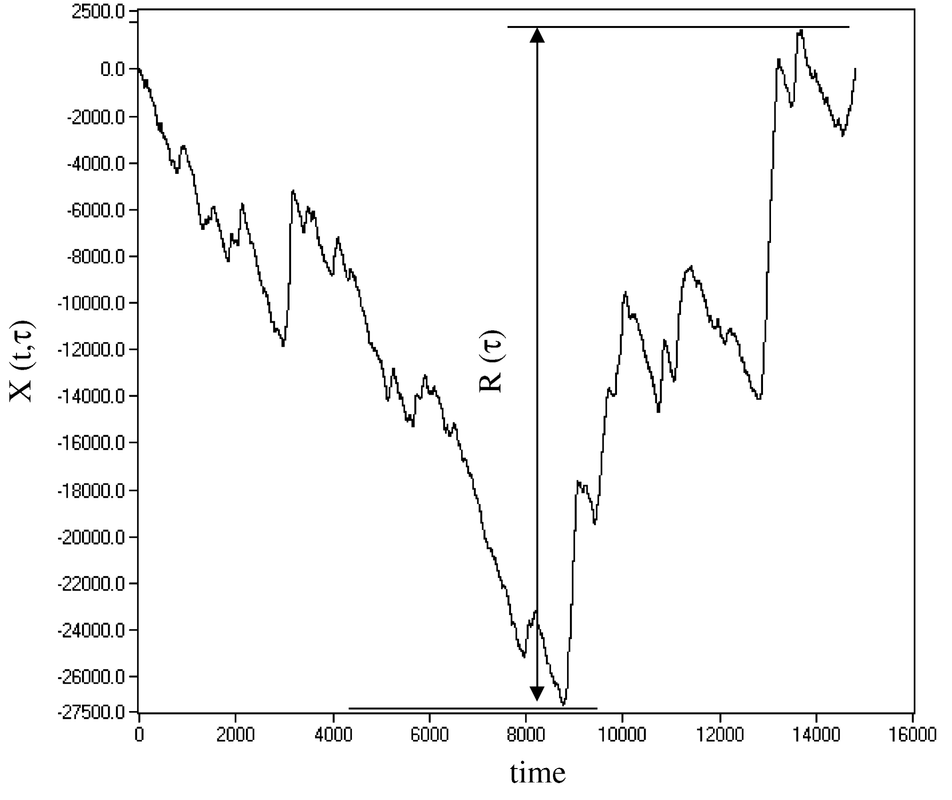

Fig. 1. The 'virtual reservoir level' $X(t, \tau)$ (see Eq. (1)), as determined from simulations of the 2d Oslo-model over 15,000 time steps. As will be seen in further analysis below, a self-similar structure is formed. The range $R(\tau)$, is also indicated in the figure.

$q(t)$ at time t,

$$X(t, \tau) = \sum_{u=1}^{t} [q(u) - \langle q \rangle_\tau] \Delta t \; , \tag{1}$$

where Δt is the time step of the measurements and

$$\langle q \rangle_\tau = \frac{1}{\tau} \sum_{t=1}^{\tau} q(t) \Delta t \tag{2}$$

denotes the average influx over the period of the experiment, τ.

Given such a water reservoir level, Hurst [3] compared the range of $X(t, \tau)$ (see also Fig. 1), defined by

$$R(\tau) = \max_{1 \leqslant t \leqslant \tau} X(t, \tau) - \min_{1 \leqslant t \leqslant \tau} X(t, \tau) \tag{3}$$

with the standard deviation of the influx to $X(t,\tau)$, which is given by the rain rate $q(t)$

$$S(\tau) = \left(\frac{1}{\tau - 1} \sum_{t=1}^{\tau} [q(t) - \langle q \rangle_\tau]^2 \right)^{1/2} . \tag{4}$$

The dimensionless ratio of $R(\tau)/S(\tau)$ was found by Hurst [3] to scale as a power law for the water levels of the Nile, $R/S(\tau) \sim \tau^H$, where he found $H \approx 0.77$ for the value of the exponent. For uncorrelated random events, the exponent can be shown to be $H = \frac{1}{2}$ [3] and subsequently this R/S method was introduced by Mandelbrot and Wallis as a general tool to characterize self-similar time series [8], using the anomalous Hurst exponent.

3. The model

The sandpile model used in the simulations is generally known as the Oslo-model [5] and belongs to the class of sandpile models with randomized updating rules, where the critical slope is dynamically adjusted by the avalanches [9]. In the original 1d version of the model, grains of sand are introduced at one end of a box. When the local slope of the pile, given by the height difference between the site and its adjacent neighbour, exceeds a similarly local threshold, the grains redistribute by letting one grain topple to the next site (from site i to site $i+1$). This is done by reducing the height at position i by one and increasing the height at position $i+1$ by one. Furthermore, at such a toppling event the local threshold is chosen anew. This procedure is iterated until all sites have reached a stable state, at which point a new grain is introduced on the left. Due to the randomization of the updating rule, the average critical slope actually self-organizes to a value intrinsic in the model. The model can easily be re-formulated in terms of the local slopes $z_i = h_i - h_{i+1}$. In that case, the rules are as follows: when $z_i > z_c$ [5]

$$\begin{aligned} z_i &\to z_i - 2 , \\ z_{i\pm1} &\to z_{i\pm1} + 1 , \\ z_c &\in \{1,2\} . \end{aligned} \tag{5}$$

In this formulation, the original sandpile nature of the model is no longer immediately obvious. However, an extension to higher dimensions becomes straightforward. Such higher dimensional versions of the model may also be useful for work on sandpiles, where the z's then stand for the gradient of the pile, but in what follows, the additional dimension will take into account properties of the atmosphere. The simplest possible extension of the toppling rules (Eq. (5)) to two dimensions encompasses the following updating rules:

$$\begin{aligned} z_{i,j} &\to z_{i,j} - 4 , \\ z_{i\pm1,j} &\to z_{i\pm1,j} + 1 , \\ z_{i,j\pm1} &\to z_{i,j\pm1} + 1 , \\ z_c &\in \{3,4\} . \end{aligned} \tag{6}$$

The updating for this two-dimensional version takes place on a line at $x = 0$, whose values are increased by one after each avalanche (i.e., $z_{0,j} \rightarrow z_{0,j}+1$). In the y-direction, periodic boundary conditions are assumed. The boundaries in the x-direction are open. This constitutes what will be referred to as the 2d Oslo-model in the following.

In the context of rainfall, this formulation of the model is more intuitive. Here, one may interpret the $z_{i,j}$ as the coarse-grained levels of water vapour (humidity) in the atmosphere, which is continuously increased at the surface of the earth due to the evaporation from the oceans (i.e., the edge of the pile—or the line at $x = 0$) and which condenses after reaching the (locally defined) dew-point, thereby influencing the humidity in the immediate surroundings. Due to the coarse-grained nature of the model, the scale on which humidity of the air packets in question increases or decreases can be macroscopic. Therefore, for instance turbulent eddies in the atmosphere may lead to an effective increase of humidity in the air packet neighbouring the condensation, on the scale of the model. The details of the microscopic process are however not important. In addition, after such a condensation event, the local dew-point is changed as well, due to the change in temperature that follows the condensation. Furthermore, the second dimension of the model should take lateral diffusion in the atmosphere into account as well. Thus, when thinking of modelling rain distributions, the x-axis would correspond to the height of the atmosphere, whereas the y-axis is the lateral direction. The main point of this paper is that with such an interpretation of the relevant parameters, the 2d Oslo-model can be treated as an effective, minimal model for rain events, where each toppling event corresponds to the condensation of water out of the atmosphere. The 'rain rate' at a given moment is then given by the number of topplings (condensation events) in one iteration and a rain event corresponds to an avalanche.

An example of the reservoir level obtained from the rain rate produced in the model can be seen in Fig. 1. It should be noted at this point that the way time is defined in these simulations differs substantially from the way it is usually done in sandpile models, but is very much in line with the different versions of time in the analysis of rain events [2]. In order to obtain the reservoir level shown in Fig. 1, every iteration of the model was taken as a time step, instead of the duration of an avalanche.

4. Results and discussion

The results of an R/S analysis averaged over 50 different runs of the 2d Oslo-model are shown in Fig. 2. In the simulations, an area of 100×100 pixels was used. Other area sizes have been simulated as well, with consistent results. As can be seen from Fig. 2, power-law scaling is obtained for almost four decades, with a Hurst exponent of $H^{\mathrm{Oslo}} = 0.80(5)$. This has to be compared to Fig. 4 of Ref. [2], where similar behaviour is found from actual rain data. In that experiment, the Hurst exponent turns out to be $H^{\mathrm{rain}} = 0.76$ [2], which is in good agreement with the behaviour of Fig. 2. Similarly, the classic work of Hurst on the water level fluctuations of the Nile on completely different time scales is in very good agreement showing an $H^{\mathrm{Nile}} = 0.77$ [3].

However, quantitative agreement of the 2d Oslo-model with the rain data from the Baltic Sea [2,8] is obtained not only in the R/S analysis. In the more classical measure

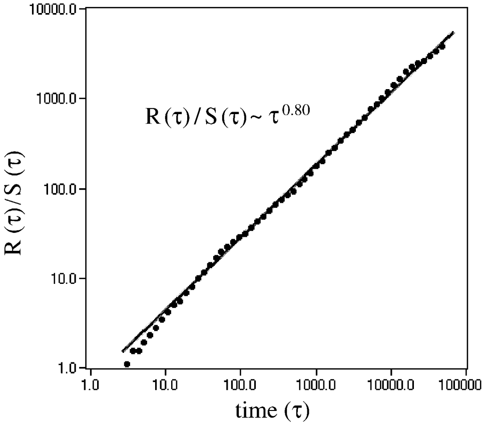

Fig. 2. Ratio R/S for simulations of the 2d Oslo-model. In the simulations an area of 100×100 pixels was run 50 consecutive times over 100,000 time steps, with about 500 avalanches passing in each run. The Hurst exponent derived from this figure is $H^{\mathrm{Oslo}} = 0.80(5)$, consistent with that determined for rainfall [2].

for SOC [4], the size distribution of avalanches, the 2d Oslo-model also gives good agreement with the experimental data on rain distribution. As can be seen from Fig. 3, the avalanche size distribution is a power law, with an exponent of $A^{\mathrm{Oslo}} = -1.40(5)$. In this figure, the size distribution of the avalanches is plotted after 5000 avalanches have passed on an area of 100×100 pixels, just as was the case for the results in Fig. 2. This result has to be compared with Fig. 1 in Ref. [2], where the rain event size distribution is shown. Again there is power-law behaviour over many decades, with an exponent of $A^{\mathrm{rain}} = -1.36$ [2], which again is in good quantitative agreement with the simulations of the Oslo-model shown above.

Another quantity determined for rainfall in Ref. [2], is the size distribution of drought durations. Due to the discreet nature of the cellular automaton used here (after at least three updates, the threshold z_c is exceeded and a toppling occurs—hence drought periods cannot last longer than three time steps), the model is not able to reproduce such a result. Nevertheless, extensions of the model are possible, taking into account the finite resolution of experiments and hence only registering rain rates (i.e., toppling numbers) bigger than a certain size. Other options include changing the updating rules, which encompasses letting a certain number of time steps pass between updates instead of updating after a full avalanche has passed. Such models have been simulated and can give behaviour which is consistent with the data in Ref. [2] for certain choices of the additional parameters. Such additions to SOC models have been done previously for an earthquake model, where power-law waiting statistics could be obtained [10]. However, in the spirit of models of SOC, I do not want to introduce further parameters, the choice of which would have to be justified on more detailed, microscopic grounds. From a similar position, Sanchez et al. [11] have recently argued that waiting time statistics present a bad measure with which to test SOC models.

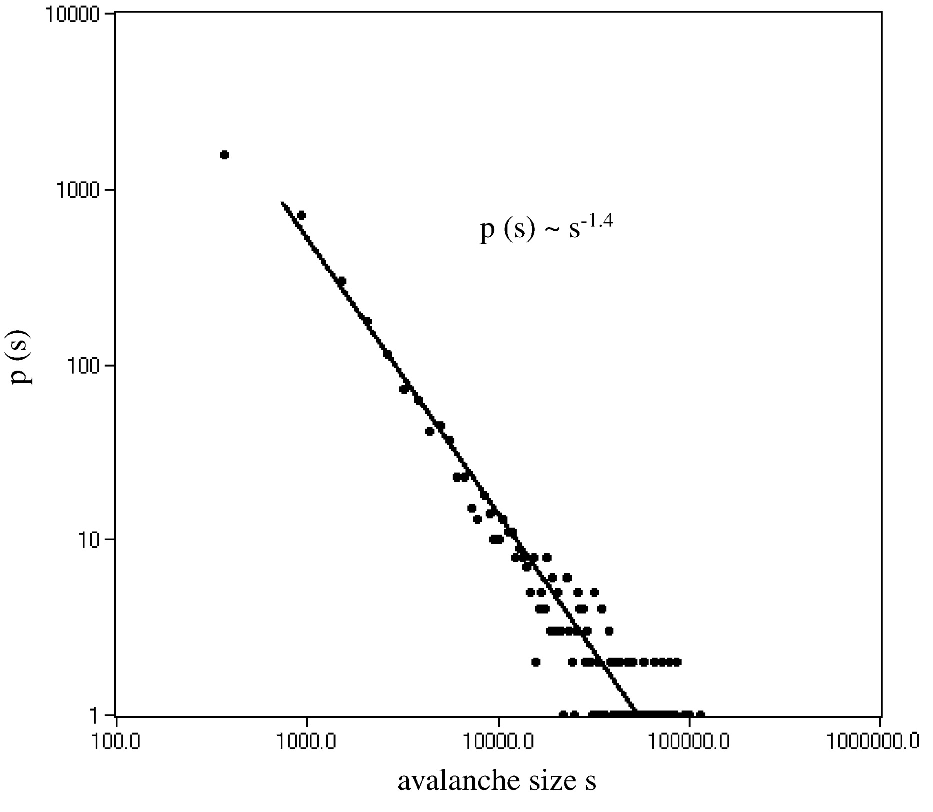

Fig. 3. Avalanche size distribution for the 2d Oslo-model. An area of 100×100 pixels was simulated during 5000 avalanches, the size distribution of which is shown. It follows a power law, with an exponent of $A^{\mathrm{Oslo}} = 1.40(5)$, which is consistent with that of the size distribution of rain events [2].

In order to test the applicability of the Oslo-model to rainfall further, it may be interesting to study the structure of clusters of activity. It is known that in the 2d Bak–Tang–Wiesenfeld (BTW) model [4], those clusters form fractal structures. In Fig. 4, some examples of activity clusters are shown for the 2d Oslo-model introduced here. From the figure it can be seen that just as in the BTW-model, the activity clusters seem to have a fractal boundary. This is intriguing, since in the present interpretation of the Oslo-model, these clusters of activity correspond to clusters where water has condensed in the atmosphere, which are more generally known as clouds. For cumulus clouds (i.e., rain clouds) it is also well known that their boundaries show fractal shapes [12]. It would be interesting to study the scaling properties of the active clusters in more detail and compare them to those observed experimentally in clouds, where a wealth of information is available [13]. However, for such a comparison to be done properly, the Oslo-model will probably have to be extended to three dimensions. Nevertheless, this can in principle be achieved easily, in analogy to the 2d extension proposed here, but will be computationally demanding for reasonable size grids.

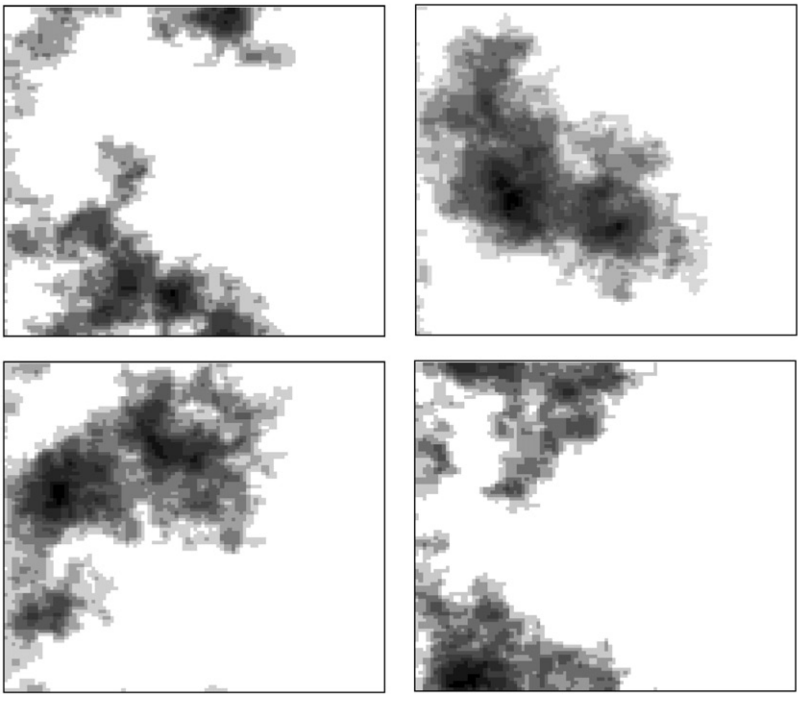

Fig. 4. Four different activity clusters in a 100×100 simulation of the 2d Oslo-model. As can be seen, these clusters have fractal boundaries, similar to clouds, which is what these clusters correspond to in the present interpretation of the model.

5. Conclusions

In conclusion, simulations of the 2d Oslo-model have been studied in terms of an R/S analysis and a Hurst exponent of $H^{\mathrm{Oslo}} = 0.80(5)$ has been found. In these simulations, each iteration has been treated as an individual time step, which is in contrast to the usual way of treating sandpile models. In the case of modelling rainfall, however, physical time and the time of rain events are different, which justifies the different treatment of time in the simulations.

Comparing the results of simulations with those of actual rainfall in the Baltic Sea, one finds that the Hurst exponent is quantitatively reproduced. Peters et al. [2] find a value of $H^{\mathrm{rain}} = 0.76$, in agreement with $H^{\mathrm{Oslo}} = 0.80(5)$ for the simulations presented here. Furthermore, there was another recent R/S analysis of rainfall distributions in Queensland, Australia, which gives similar results [14]. Moreover in terms of the 'avalanche size distribution', which in the case of rainfall corresponds to the size distribution of rain events, agreement between the model and the experimental situation is found. In the simulations the exponent of the power-law distribution of avalanches is $A^{\mathrm{Oslo}} = -1.40(5)$, whereas in the experiment a value of $A^{\mathrm{rain}} = -1.36$ was obtained [2].

This means that there is quantitative evidence from a model known to exhibit SOC that the distribution of rain events may be reproduced from such a model, indicating that the governing processes in the atmosphere may be in a SOC state. Further possibilities for quantitative comparisons between such SOC models and meteorological data in terms of cloud shapes are pointed out as well. Moreover, it should be noted that R/S analyses have not been restricted to rainfall in the past but to many other systems including charge transport in plasmas, where also a Hurst exponent of $H^{\mathrm{plasma}} = 0.77$ was found experimentally [15]. In addition, the temporal variation of the rotation of the Sun [16], as well as the variation of Sun-spots [17] show behaviour consistent with the simulations giving $H^{\mathrm{Sun}} = 0.83$. Finally, C-14 radiocarbon abundance variation in the atmosphere over the last 3000 years give the same behaviour with $H^{\mathrm{C\text{-}14}} = 0.84$ [18]. Even though all of these phenomena seem to be far removed from the distribution of rainfall (and each other), charge transport in plasmas, as well as Sun-spot variations are both concerned with turbulent transport, which is also the case for the rainfall distribution. Furthermore, radiocarbon abundance may be directly related to Sun-spot variation via solar activity and the corresponding cosmic radiation. This indicates that simple sandpile models can actually give *quantitative* agreement with a great variety of different systems.

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